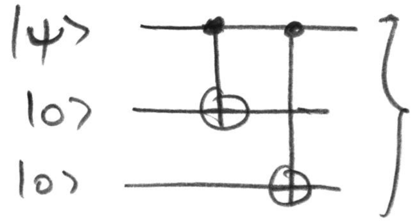
i.e.: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{encoding}} |\hat{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$

Encoding circuit:



$|\hat{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$

Consider bit flip error $|\hat{\psi}\rangle \longmapsto X_i|\hat{\psi}\rangle$ on qubit i.

Can we correct for one bit flip error on unknown qubit?

Problem: Meas. all qubits destroys q. info!

$\Rightarrow$ Need meas. which only returns info about location of error (indep. of encoded state $|\psi\rangle$).

Define "syndrome measurement" with projectors:

"no flip": $P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$

"1st qubit flipped": $P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$

"2nd qubit flipped": $P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$

"3rd qubit flipped": $P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$

Measuring $\{P_\alpha\}$ reveals only 2 bits of information

$\Rightarrow$ one qubit untouched,

By inspection: info. obtained $\equiv$ location of flip, e.g.

$$\alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{qubit } 1]{\text{flip of}} \alpha|100\rangle + \beta|011\rangle$$

$\Rightarrow$ meas. returns P_1, post meas. state

$$\alpha|100\rangle + \beta|011\rangle \xrightarrow[\text{qubit } 1]{\text{recovery: flip}} \alpha|000\rangle + \beta|111\rangle$$

Works for any single or no flipped qubit, & all states $|\psi\rangle$.

(Linearity: also works for parts of large entangled state.)

What about continuous errors, such as

$$|\hat{\psi}\rangle \longmapsto e^{i\vartheta X_i}|\hat{\psi}\rangle = (\cos\vartheta\, \mathbb{1} + i\sin\vartheta\, X_i)|\hat{\psi}\rangle \ ?$$

$$|\hat{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{qubit } 1]{\text{error, eg}} \cos\vartheta \underbrace{(\alpha|000\rangle + \beta|111\rangle)}_{\text{syndrome } P_0} + i\sin\vartheta \underbrace{(\alpha|100\rangle + \beta|011\rangle)}_{\text{syndrome } P_1}$$

Syndrome meas. collapses state to:

$p = \cos^2\vartheta$: $P_0 \Rightarrow \alpha|000\rangle + \beta|111\rangle$, no correction ✓

$p = \sin^2\vartheta$: $P_1 \Rightarrow \alpha|100\rangle + \beta|011\rangle$, flip bit 1 ✓

Meas. of error syndrome P_i collapses error onto "digital" error — no error or bit flip $\Rightarrow$ sufficient to study discrete errors!

What about Z errors?

$$\alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{on some qubit}]{\text{Z error}} \alpha|000\rangle - \beta|111\rangle$$

This is still in <u>code space</u> (i.e., space of valid $|\psi\rangle$)

$\Rightarrow$ error <u>not detectable</u>, but it <u>has changed $|\psi\rangle$</u>!

$\Rightarrow$ 3-qubit bit flip code cannot protect against "phase flip" Z.

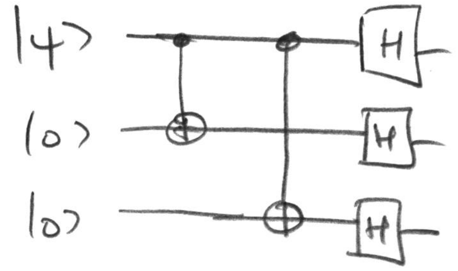
<u>In fact:</u> Phase flip Z_i acts as logical Z on the encoded qubit.

<u>6) 3-qubit phase flip code</u>

We have $Z|+\rangle = |-\rangle$, $Z|-\rangle = |+\rangle$

Z error $\hat{=}$ bit flip error in $|\pm\rangle$-basis.

Encoding $|\hat{0}\rangle = |{+}{+}{+}\rangle$, $|\hat{1}\rangle = |{-}{-}{-}\rangle$ will protect against Z errors!



Syndrome meas. $\tilde{P}_\alpha := H^{\otimes 3} P_\alpha H^{\otimes 3}$, recovery $H X_\alpha H = Z_\alpha$.

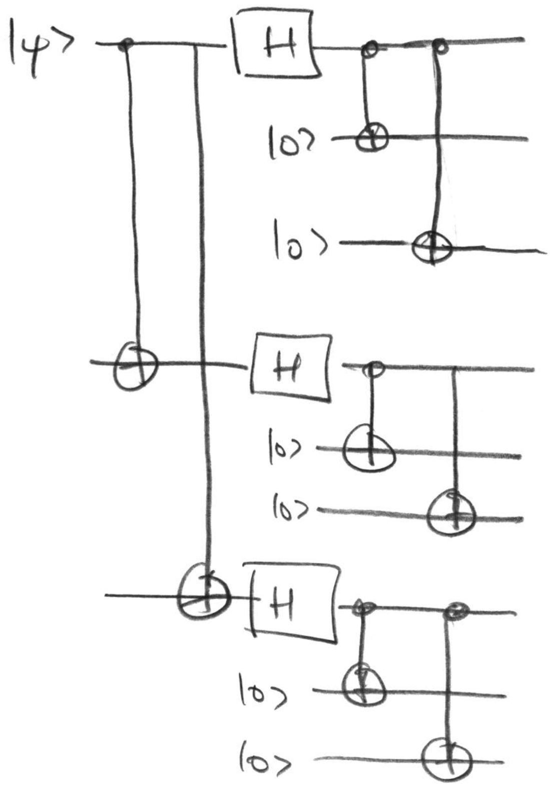
<u>Problem</u>: Now, no protection against bit-flip errors.
(and X_i acts as X on logical qubit)

<u>V.2, The 9-qubit Shor code</u>

<u>Solution</u>: Concatenate (= nest) 3-qubit bit flip with 3-qubit phase flip code!

$$|0\rangle \longmapsto |+\rangle|+\rangle|+\rangle \longmapsto \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \longmapsto |-\rangle|-\rangle|-\rangle \longmapsto \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$



9-qubit Shor code

Can correct <u>any</u> single-qubit Pauli:

(i) X_i error is corrected at "inner" layer.

(ii) Z_i error $\equiv$ <u>logical</u> error on "outer" qubit

$\Longrightarrow$ $Z_{block(i)}$ error on "outer" code (phase-flip)

$\Rightarrow$ correctable!

(iii) $Y_i \propto X_i Z_i$: X_i & Z_i corrected independently.

V.3 Quantum Error Correction Conditions

Quantum Error Correcting Code (QECC):

Defined by code space C (elements: "codewords").
Choose basis $|\hat{i}\rangle$.

Noise model: CPTP map

$$\mathcal{E}(\rho) = \sum E_\alpha \rho E_\alpha^\dagger \ ; \ \sum E_\alpha^\dagger E_\alpha = \mathbb{1}.$$

Recovery procedure: Measurement + recovery

$\Leftrightarrow$ another CP map $\mathcal{R}$.

Require that $\mathcal{R}(\mathcal{E}(\rho)) = \rho$ for all $\rho = |\psi\rangle\langle\psi|, |\psi\rangle \in C$

Under which conditions on C & $\mathcal{E}$ does such an $\mathcal{R}$ exist?

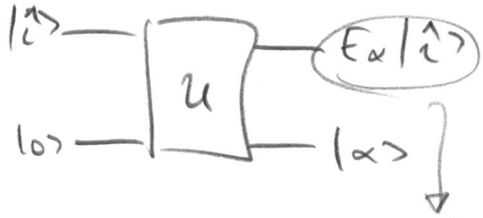
Necessary conditions:

(i) Environment carries no information about ρ for any $\rho = \sum \rho^{ij} |\hat{i}\rangle\langle\hat{j}|$ in C:

linearity $\Longrightarrow$ $\underbrace{\langle \hat{i} | E_\alpha^\dagger E_\alpha | \hat{i} \rangle}_{= \text{prob}(\alpha)} = c_\alpha$ (indep. of $\hat{i}$)

... recall:

$$|\hat{i}\rangle\langle\hat{i}| \mapsto \mathcal{E}(|\hat{i}\rangle\langle\hat{i}|) \Longleftrightarrow$$



Ch. II: Stinespring dilation!

output for env. state $|\alpha\rangle$.

(ii) orthogonal states must <u>remain orthogonal</u> (R cannot make states more orthogonal!)

$$\mathcal{E}(|\hat{i}\rangle\langle\hat{i}|) \perp \mathcal{E}(|\hat{j}\rangle\langle\hat{j}|) \quad \text{for } \langle\hat{i}|\hat{j}\rangle = 0$$

orth. support!

i.e.
$$\delta_{ij} \propto \mathrm{tr}\left(\mathcal{E}(|\hat{i}\rangle\langle\hat{i}|)\,\mathcal{E}(|\hat{j}\rangle\langle\hat{j}|)\right)$$

$$= \sum_{\alpha\beta} \mathrm{tr}\left(E_\alpha |\hat{i}\rangle\langle\hat{i}| E_\alpha^\dagger E_\beta |\hat{j}\rangle\langle\hat{j}| E_\beta^\dagger\right)$$

$$= \sum_{\alpha\beta} \left|\langle\hat{i}|E_\alpha^\dagger E_\beta|\hat{j}\rangle\right|^2$$

(i) + (ii) $\Longrightarrow$
$$\boxed{\langle\hat{i}|E_\alpha^\dagger E_\beta|\hat{j}\rangle = c_{\alpha\beta}\,\delta_{ij}} \quad (c_{\alpha\beta} = c_{\beta\alpha}^*)$$

<u>Quantum Error Correction Condition</u>

This is also sufficient:

Use gauge deg. of freedom:

$$\sum E_\alpha \rho E_\alpha^+ = \sum F_\beta \rho F_\beta^+ \quad \text{w/} \quad F_\beta = \sum \underset{\uparrow \, \boxed{\text{isometry}}}{V_{\beta\alpha}} E_\alpha$$

to choose F_β s.th. $c_{\alpha\beta}$ becomes diagonal,

$$\langle \hat{\imath} | F_\alpha^+ F_\beta | \hat{\jmath} \rangle = \lambda_\alpha \delta_{\alpha\beta} \delta_{ij}$$

i.e.: Different α can be distinguished by measurement and undone:

$$\underbrace{\left(\frac{1}{\lambda_\beta} \sum_{\hat{\imath}} |\hat{\imath}\rangle\langle\hat{\imath}| F_\beta^+ \right)}_{\text{Kraus op. of recovery map}} F_\alpha | \hat{\jmath} \rangle = \delta_{\alpha\beta} | \hat{\jmath} \rangle$$

$$\langle\hat{\imath}| F_\beta^+ F_\alpha | \hat{\jmath} \rangle = \lambda_\alpha \delta_{\alpha\beta} \delta_{ij}$$

Note: For a single-qubit error,

$$E_\alpha = \sum_{k,s} \omega_{\alpha,k,s} \sigma_s^k$$

Pauli σ^k on qubit s

i.e: $\langle \hat{\imath} | (\sigma_k^s)^+ \sigma_\ell^r | \hat{\jmath} \rangle \propto \delta_{ij} \Longrightarrow \langle \hat{\imath} | E_\alpha^+ E_\beta | \hat{\jmath} \rangle \propto \delta_{ij}$.

i.e.: Err. Corr. Cond. for Paulis $\Rightarrow$ err. corr. cond. for any single-qubit error!

In particular: Robust to depolarizing channel

$$\mathcal{E}(\rho) = p\rho + \frac{(1-p)}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$

on every qubit $\Rightarrow$ robust to any 1-qubit noise!

... and similar for k-qubit errors & k-fold Pauli products!

Basic properties of QECC:

Focus: "binary codes": encode k qubits in n > k qubits.

"Distance" d: Smallest # of Paulis ($\neq I$)

in $E_\alpha = P \otimes I \otimes \ldots \otimes P \otimes \ldots$ s.th.

$$\langle \hat{i} | E_\alpha | \hat{j} \rangle \neq \lambda_\alpha \delta_{ij}$$

Notation: $\boxed{[\![n, k, d]\!]\text{-code}}$

phys. qubits

encoded qubits

distance

How many 1-qubit errors t can a distance d code correct?

For E_α, E_β with $\leq t$ Paulis:

$$\langle \hat{i} | \underbrace{E_\alpha^\dagger E_\beta}_{\leq 2t \text{ Paulis}} | \hat{j} \rangle \overset{?}{=} c_{\alpha\beta}\, \delta_{ij} \iff \boxed{2t+1 \leq d}$$

i.e.: For $d=3$, we can correct all 1-qubit errors.

Note: If location of errors is known, $E_\alpha^\dagger E_\beta$ has $\leq t$ Paulis

$$\implies \boxed{t+1 \leq d}$$

or: code can correct t errors in unknown locations

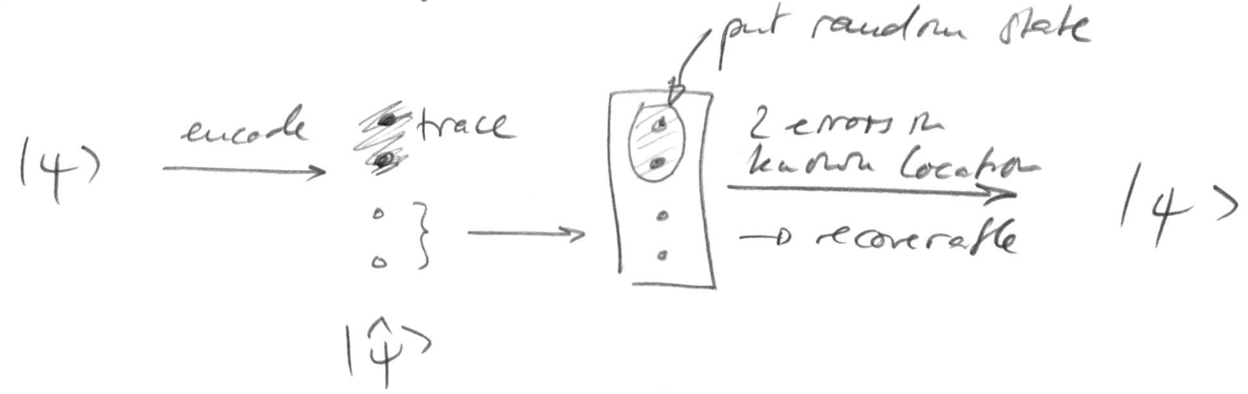
$\iff$ t can correct $2t$ errors in known locations

Are there constraints on $[[n,k,d]]$?

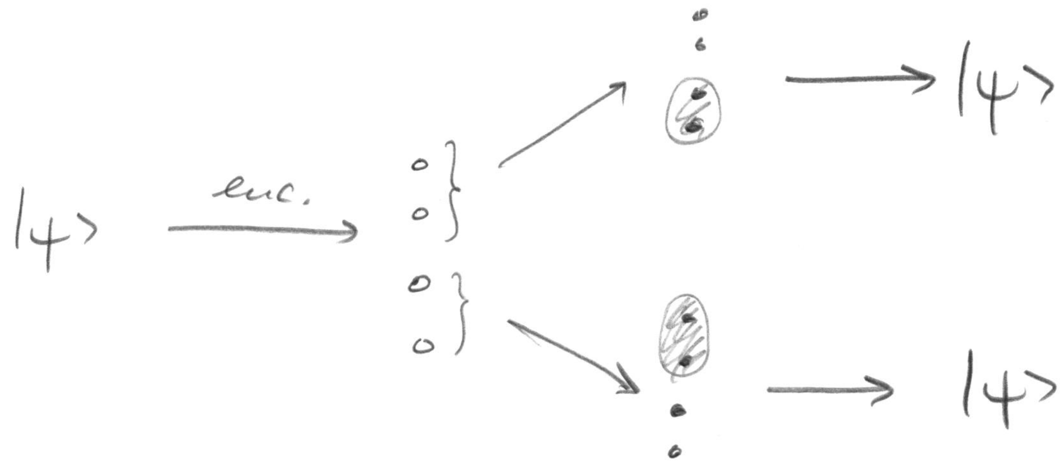
Simplest case: $k=1$, $d=3$. What is minimal n?

Claim: $n \geq 5$!

Proof via no-cloning:



So...



↯ violates no-cloning!

[[5,1,3]] code optimal.

Will see: it exists!