IV. Quantum Computation

IV.1 The circuit model

a) Classical computation

Use of class. computers:

Solve problems $\equiv$ compute functions

$$f : \{0,1\}^n \longrightarrow \{0,1\}^m$$

$$\underline{x} = (x_1, \dots, x_n) \longmapsto f(x_1, \dots, x_n) = \underline{y} = (y_1, \dots, y_m)$$

f depends on problem, $\underline{x}$ encodes instance of problem.

E.g.: Multiplication $(a, b) \longmapsto a \cdot b$

$$\underline{x} = \underbrace{(\underline{x}_1, \underline{x}_2)}_{\text{enc. in binary}} \Longrightarrow f(\underline{x}) = \underbrace{\underline{x}_1 \cdot \underline{x}_2}_{\text{enc. in binary}}$$

Factorization: $\underline{x}$: integer; $f(\underline{x})$: list of prime factors (w/ suitable encoding)

Each problem is encoded by a <u>family</u> of functions

$$f \equiv f^{(n)} : \{0,1\}^n \longrightarrow \{0,1\}^m \; ; \; m = \mathrm{poly}(n); \; n \in \mathbb{N}.$$

<u>What ingredients</u> do we need to compute a general function f?

(i) $f : \{0,1\}^n \longrightarrow \{0,1\}^m$

$$f(\underline{x}) = (f_1(\underline{x}), f_2(\underline{x}), \dots, f_m(\underline{x}))$$

$$f_k : \{0,1\}^n \longrightarrow \{0,1\}$$

$\Rightarrow$ can focus on <u>boolean functions</u> $f : \{0,1\}^n \longrightarrow \{0,1\}$.

(ii) Let $L = \{ \underline{y} \mid f(\underline{y}) = 1 \} = \{ \underline{y}^1, \underline{y}^2, \dots, \underline{y}^\ell \}$

Define $g_{\underline{y}}(\underline{x}) = \begin{cases} 0 \; ; \; \underline{x} \neq \underline{y} \\ 1 \; ; \; \underline{x} = \underline{y} \end{cases}$

Then, $f(\underline{x}) = g_{\underline{y}^1}(\underline{x}) \vee g_{\underline{y}^2}(\underline{x}) \vee \dots \vee g_{\underline{y}^\ell}(\underline{x})$

"$\vee$" $\equiv$ "logical or":

$$0 \vee 0 = 0$$
$$0 \vee 1 = 1$$
$$1 \vee 0 = 1$$
$$1 \vee 1 = 1$$

<u>associative:</u> $a \vee b \vee c := (a \vee b) \vee c = a \vee (b \vee c)$

(iii) $g_{\underline{y}}(\underline{x}) = \underbrace{(y_1 = x_1)}_{\substack{1 \text{ if } y_1 = x_1 \\ 0 \text{ if } y_1 \neq x_1}} \wedge (y_2 = x_2) \wedge \ldots$

"$\wedge$" $\equiv$ "logical and": $1 \wedge 1 = 1$, else 0.

(iv) $$(y_i = x_i) = \begin{cases} x_i, & \text{if } y_i = 1 \\ \neg x_i, & \text{if } y_i = 0 \end{cases}$$

"$\neg$" $\equiv$ "logical not": $\neg 1 = 0$, $\neg 0 = 1$.

$\Rightarrow$ $f(\underline{x})$ can be built from "and", "or", "not" gates, and <u>copy gates</u> $x \mapsto (x, x)$.

<u>"Universal gate set"</u>

(<u>Note</u>: In fact, "nand" $\neg(x \wedge y)$ or "nor" $\neg(x \vee y)$ are by themselves already universal together w/ copy.)

<u>"Circuit model"</u> of <u>computation</u>: $f \equiv f^{(n)}$ can be built from a simple universal gate set w/out loops (i.e., gates are applied sequentially).
(Technical point: The circuit family for $n \in \mathbb{N}$ must be generatable in an efficient way, e.g. a simple & fast program.)

The hardness of a problem is meas. by number $K(n)$ gates needed to compute $f^{(n)}$. ($\hat{=}$ # of time steps).

Can distinguish different regimes:

$K(n) \sim \mathrm{poly}(n)$: efficiently solvable (class P)

$K(n) \gtrsim \mathrm{poly}(n)$, e.g. $K(n) \sim \exp(n)$: hard problem

Are there more hard or easy problems?

random $f : \{0,1\}^n \longrightarrow \{0,1\}$

# of possible f: $2^{(2^n)}$

2 ← 0 or 1 for each input

2^n ← # of inputs

But there are only $c^{\mathrm{poly}(n)}$ circuits of length poly(n)!

c ← # univ. gates

$\Longrightarrow$ most f cannot be computed efficiently!

Does comp. power (= what is easy) dep. on gate set?

$\longrightarrow$ No. By def., any univ. gate set can simulate any other gate set of few-qubit gates w/ constant overhead!

Are there other models of computation?

- CPU + RAM
- parallel computer
- "Turing machine" (tape + head ≡ "linear RAM")
- cellular automata

⋮

But: All known "reasonable" models of computation can simulate each other w/ poly(n) overhead ⇒ same comput. power!

Church-Turing-Thesis: All reasonable models of computation have the same computational power.

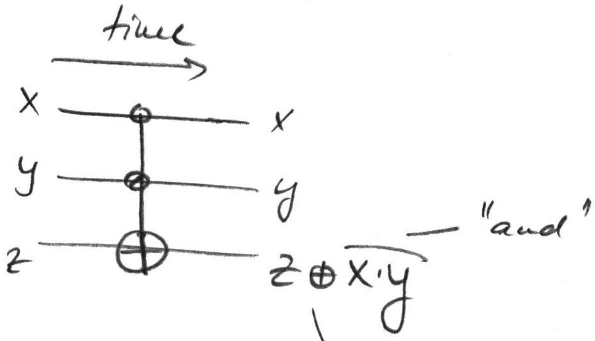
We will use circuit model for quantum computer: Gates become Unitaries!

But: Unitaries reversible ⟷ class. gates irreversible

So... can we even fit class. computation into that?

Yes! – Class. computation can be <u>turned reversible</u>:

<u>Toffoli gate:</u>



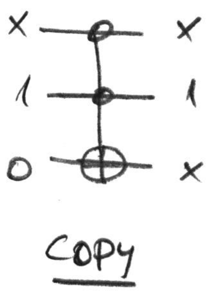
"XOR": $0 \oplus 0 = 0$
$1 \oplus 1 = 0$
$0 \oplus 1 = 1$
$1 \oplus 0 = 1$

(= addition mod 2)

→ reversible!

→ can simulate and, or, not, copy using ancillas,

e.g.:



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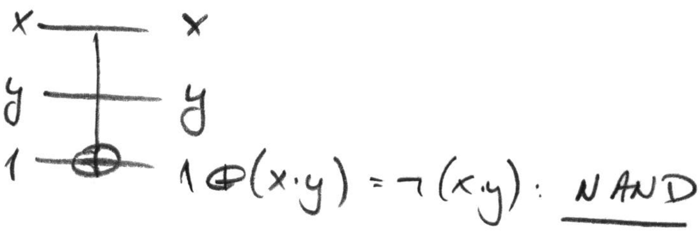


⇒ reversible univ. gate set (using ancillas)

Any $f(\underline{x})$ can be computed reversibly:

$$f^r(\underline{x}, y) = (\underline{x}, f(\underline{x}) \oplus y)$$

bitwise XOR?

(Possible w/ ancillas w/ ess. the same circuit.

<u>Idea</u>: Comp. f rev. w/ ancillas, XOR result w/ y,

and "un-compute" everything, i.e. reset ancillas.

Can be optimized to use few ancillas → Preskill's notes.)

⇒ Everything can be computed reversibly.

<u>But</u> 3-bit gate needed!

<u>b) Quantum Circuits</u>

Model of quantum computation – circuit model:

- System consists of qubits: tensor prod. structure
- Universal gate set $S = \{U_1, \dots, U_k\}$ of "small" (i.e., few-qubit) gates
- build circuits
- Input: <u>Classical</u> input $|x_1\rangle|x_2\rangle\dots|x_n\rangle \equiv |x_1, x_2, \dots\rangle$ in <u>computational basis</u> $\{|0\rangle, |1\rangle\}$, and possibly <u>ancillas</u> in $|0\rangle$ state.
- Output: Measure some (or all) qubits at end in comp. basis

(<u>Notes</u>: ◦ Other meas., e.g. POVM, can be simulated w/in model.
◦ Meas. at earlier time can be always postponed.)

Which gate set should we choose?

- Continuum of gates: much more rich!
- Different notions of universality
  - exact universality: any n-qubit gate U can be realized exactly
  - approximate universality: any n-qubit gate U can be approximated by gate set well.
    (Solovay-Kitaev Thm: ε-approx. of n-qubit gate requires $O(\mathrm{poly}(\log(1/\varepsilon)))$ gates.)
- Turns out: 1 & 2-qubit gates alone univ. ($\longleftrightarrow$ classical: 3)
- Examples of approx. univ. gate sets
  - any random 2-qubit gate (!)
  - more later, after introducing some std. gates.

- Our exact universal gate set:

(i) 1-qubit rotations about X & Z axis:

$$R_X(\phi) = e^{-iX\phi/2} \; ; \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \; X^2 = I$$

$$R_Z(\phi) = e^{-iZ\phi/2} \; ; \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \; Z^2 = I$$

If $\Pi^2 = I$: $e^{-i\Pi\phi/2} = \cos\phi/2 \, I - i \sin\phi/2 \, \Pi$

$$\Rightarrow \quad R_x(\phi) = \begin{pmatrix} \cos\phi/2 & -i\sin\phi/2 \\ -i\sin\phi/2 & \cos\phi/2 \end{pmatrix}$$

$$R_z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$$

Can also be interpreted as rotation on <u>Bloch sphere</u>, i.e. in $O(3) \cong SU(2)/\mathbb{Z}_2$, about X & Z by ϕ.

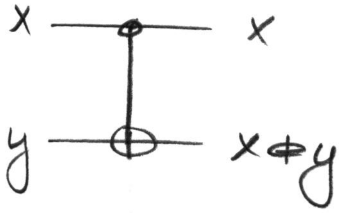
R_x & R_z can generate <u>any rotation</u> in $SU(2)/\mathbb{Z}_2 \cong O(3)$ (→ Euler angles!)

$$U = R_x(\alpha) R_z(\beta) R_x(\gamma) \; : \text{ all of } SU(2).$$

(ii) one 2-qubit gate (almost all would do!)

Typ. we choose "controlled-NOT" ≡ CNOT

CNOT =



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CNOT flips y iff x = 1: classical gate!

Can show: These gate set can create any n-qubit U **exactly** (of course not efficiently, e.g. by parameter counting).

Some important gates + identities (→ HW!)

Hadamard gate: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$; $H = H^\dagger$, $H^2 = I$

$$H R_x(\phi) H = R_z(\phi)$$

$$H R_z(\phi) H = R_x(\phi)$$

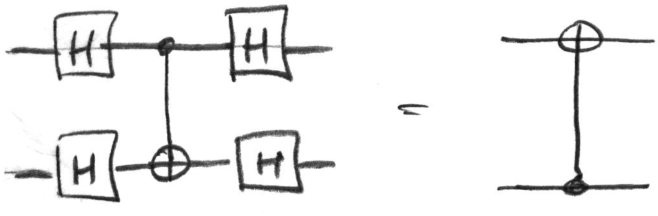
Graphical notation:

$(X = R_x(\pi),\ Z = R_z(\pi))$

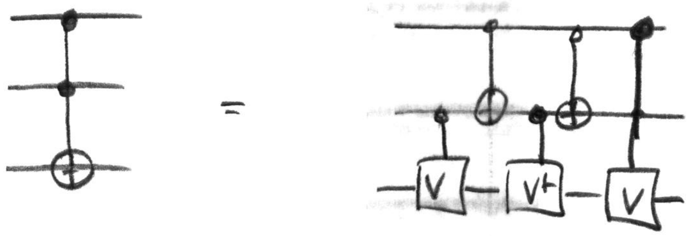
$= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$

"Controlled-Z", CZ

"Controlled phase", CPHASE



Toffoli:

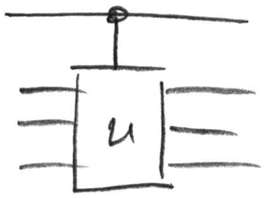


with $V = \frac{1-i}{2}(I + iX)$, and

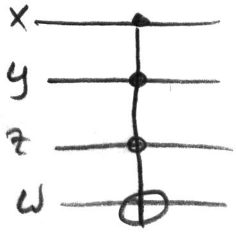
$= \left(\begin{array}{c|c} \begin{matrix}1 & \\ & 1\end{matrix} & 0 \\ \hline 0 & V \end{array}\right)$

"controlled-V"

If we can build a classical U, we can also build "controlled-U":



just replace every Toffoli (class. universal!) by Toffoli w/ 3 controls:



(flip w iff $x=y=z=1$)

Can be built from normal Toffoli ($\to$ HW)

Some more approx. univ. gate sets:

- CNOT + 2 random 1-qubit gates
- CNOT + H + $T \equiv R_Z(\pi/4)$ ("$\pi/8$ gate")

IV.2. Oracle-based algorithms

a) The Deutsch algorithm

Consider $f: \{0,1\} \to \{0,1\}$

Let f be "very hard to compute" (e.g., long circuit)

Want to know: Is $f(0) \overset{?}{=} f(1)$?

How often do we have to evaluate f (= run circuit)?

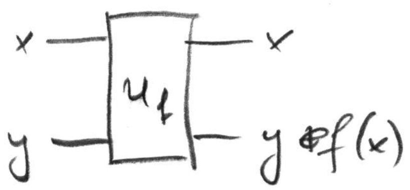
(We regard f as "black box" = "oracle":
how many queries to oracle?)

Classically: 2 queries: $f(0), f(1)$.

Can quantum mechanics do better?

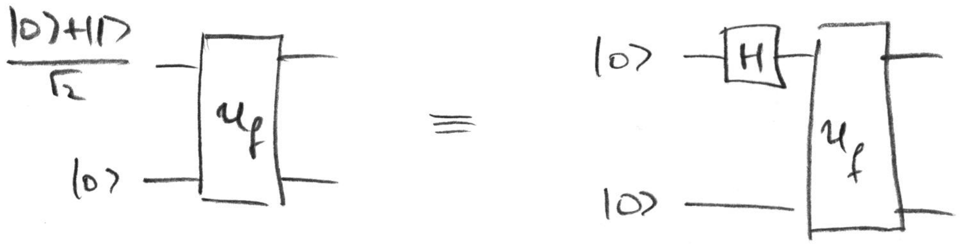
Consider reversible implementation of f:

$$f^R : (x,y) \mapsto (x, y \oplus f(x))$$



$$|x\rangle|y\rangle \mapsto |x\rangle|y \oplus f(x)\rangle$$

Try to input superpositions?



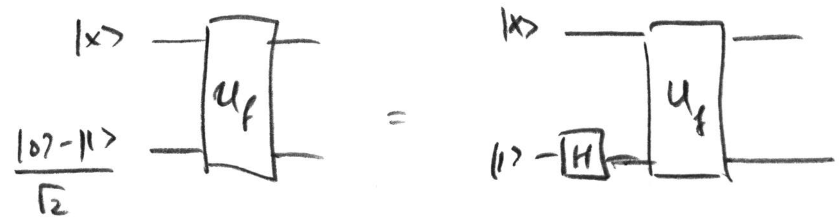
$$\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|0\rangle\right) \xmapsto{\;U_f\;} \frac{1}{\sqrt{2}}\left(|0\rangle|f(0)\rangle + |1\rangle|f(1)\rangle\right)$$

$\longrightarrow$ Have evaluated f on <u>both inputs</u>!

But: how can we <u>extract</u> relevant <u>information</u>?

- Meas. qubit 1: collapse superposition!
- Meas. qubit 2: ?

Consider instead

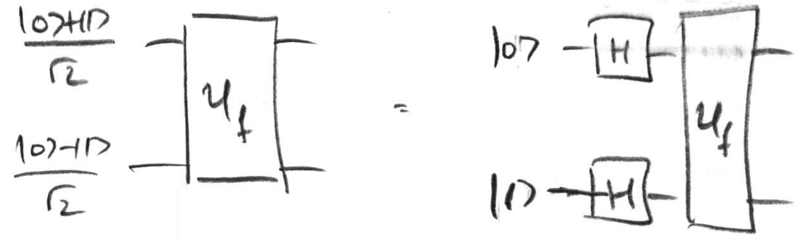


$$|x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \xmapsto{\;U_f\;} |x\rangle\left(\frac{|f(x)\rangle - |1\oplus f(x)\rangle}{\sqrt{2}}\right) =$$

$$= \left\{\begin{array}{ll} f(x)=0: & |x\rangle\,\frac{|0\rangle-|1\rangle}{\sqrt{2}} \\[2ex] f(x)=1: & |x\rangle\,\frac{|1\rangle-|0\rangle}{\sqrt{2}} \end{array}\right\} = |x\rangle\cdot\left[(-1)^{f(x)}\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$$

$$= (-1)^{f(x)}\,|x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

Combine:



$$|0\rangle|1\rangle \xmapsto{H\otimes H} \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \xmapsto{U_f} \frac{1}{\sqrt{2}}\left((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

→ no entanglement created (!)

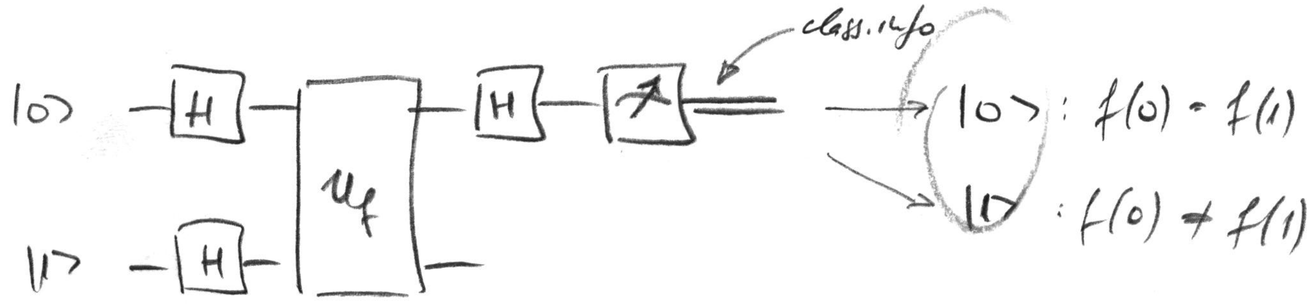
→ 2nd qubit unchanged (!!)

→ 1st qubit gets phase $(-1)^{f(x)}$

⇒ "phase kick-back" technique.

$$\Rightarrow \text{1st qubit } = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \Rightarrow f(0)=f(1)$$

$$= \frac{|0\rangle-|1\rangle}{\sqrt{2}} \Rightarrow f(0)\neq f(1)$$



<u>One application</u> of U_f sufficient ⇒ speed-up w.r.t. classical algorithm!

<u>Note</u>: 2nd qubit never measured (& contains no info!)

<u>Main ideas/points</u>:

- Use input $\sum |x\rangle$ to <u>evaluate f on all inputs</u> simult.
- Need way to read out relevant info!

6) The Deutsch-Jozsa algorithm

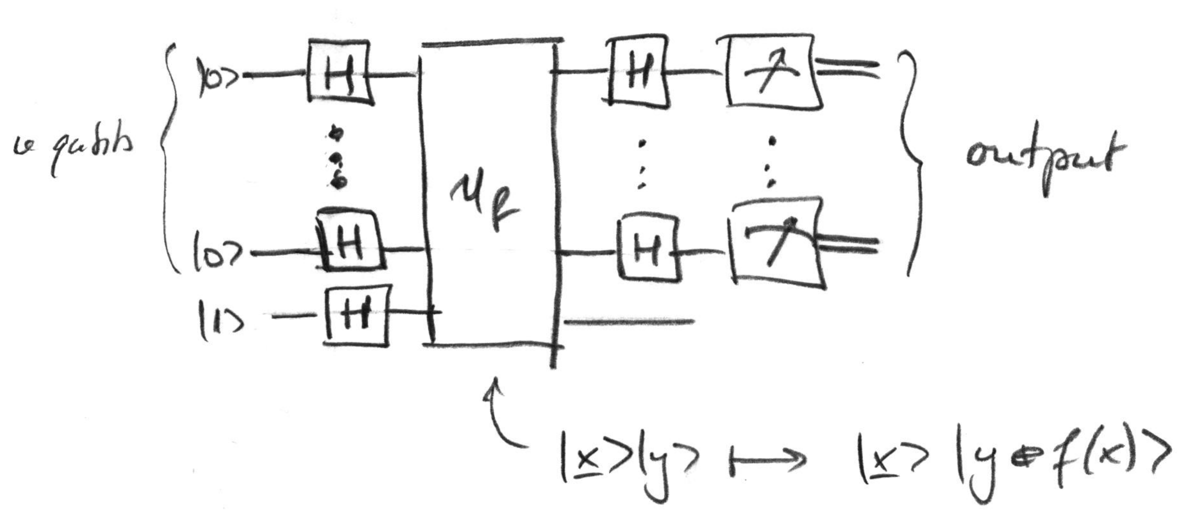
Consider $f: \{0,1\}^n \to \{0,1\}$ w/ <u>promise</u> that

<u>either</u> $f(\underline{x}) = c \quad \forall \underline{x}$ ("f <u>constant</u>")

<u>or</u> $|\{\underline{x} \mid f(\underline{x})=0\}| = |\{\underline{x} \mid f(\underline{x})=1\}|$ ("f <u>balanced</u>")

Want to know: <u>Is f constant or balanced</u>?

Use same idea: Input $\sum |\underline{x}\rangle$ and $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$:



Before analyzing circuit: What is <u>action of $H^{\otimes n}$</u>? (99)

$$H : |x\rangle \longmapsto \frac{1}{\sqrt{2}} \sum (-1)^{xy} |y\rangle$$

$$H^{\otimes n} : |x_1, \dots, x_n\rangle \longmapsto \frac{1}{\sqrt{2^n}} \sum (-1)^{x_1 y_1} (-1)^{x_2 y_2} \cdot \dots \cdot |y_1, \dots, y_n\rangle$$

or: $$|\underline{x}\rangle \longmapsto \frac{1}{\sqrt{2^n}} \sum (-1)^{\underline{x}\cdot\underline{y}} |\underline{y}\rangle$$

where $\underline{x}\cdot\underline{y} := x_1 y_1 \oplus x_2 y_2 \oplus \dots \oplus x_n y_n$,

addition mod 2

i.e. scalar prod. mod 2.

Analyze circuit:

we omit normalization!

$$|\underline{0}\rangle|1\rangle \xrightarrow{H^{\otimes n}\otimes H} \left(\sum |\underline{x}\rangle\right)\left(|0\rangle - |1\rangle\right)$$

$$\xrightarrow{U_f} \left(\sum_{\underline{x}} (-1)^{f(\underline{x})} |\underline{x}\rangle\right)\left(|0\rangle - |1\rangle\right)$$

$$\xrightarrow{H^{\otimes n}\otimes I} \Big(\sum_{\underline{y}} \underbrace{\sum_{\underline{x}} (-1)^{f(\underline{x}) + \underline{x}\cdot\underline{y}}}_{\circledast} |\underline{y}\rangle\Big)\left(|0\rangle - |1\rangle\right)$$

const!

<u>f constant</u>: $\circledast = (-1)^{f(\underline{x})} \cdot \underbrace{\sum_{\underline{x}} (-1)^{\underline{x}\cdot\underline{y}}}_{= \delta_{\underline{y},\underline{0}}} = (-1)^{f(\underline{x})} \cdot \delta_{\underline{y},\underline{0}}$

<u>f balanced</u>: For $\underline{y} = \underline{0}$, $\circledast = \sum_{\underline{x}} (-1)^{f(\underline{x}) + \underline{x}\cdot\underline{0}} = \sum_{\underline{x}} (-1)^{f(\underline{x})} = 0$

Thus: output is $y = \underline{0}$ $\Rightarrow$ f constant

output is $y \neq \underline{0}$ $\Rightarrow$ f balanced

$\Rightarrow$ Unambiguous discrimination w/ one evaluation of f!

What is speed-up w.r.t. classical?

- Quantum: 1 use of f.
- Classical: Worst case, we need to test $2^n/2 + 1$ values of f $\Rightarrow$ const. vs. exponential!

But: If we only want right answer w/ high probability $p = 1 - \varepsilon$, then for k queries to f

$$p_{error} \approx \underbrace{2 \cdot \left(\frac{1}{2}\right)^k}_{\text{approx. prob. for } k \text{ eq. outcomes for balanced } f, \ k \ll 2^n} \stackrel{!}{=} \varepsilon$$

$\Rightarrow$ $k \sim \log(1/\varepsilon)$.

$\Rightarrow$ Much smaller speed-up vs. probabilistic classical algorithm (even for exp. small error, $k \sim n$).