Is ρ_A uniquely determined by

$$\mathrm{tr}[M \rho_A] = {}_{AB}\langle\psi| \, M \otimes \mathbb{1} \, |\psi\rangle_{AB} \; ?$$

<u>Yes</u>: $\mathrm{tr}[X^\dagger Y]$ is scalar product, & overlap of ρ_A w/ all herm. M determines herm. part of ρ_A <u>entirely</u>!

(<u>Consequence</u>: All numbers in ρ_A meaningful $\Rightarrow$ no phase ambiguity!)

What <u>is</u> ρ_A for pure state $|\phi\rangle_A$?

$$\langle\phi|M|\phi\rangle = \mathrm{tr}\big[\langle\phi|M|\phi\rangle\big] = \mathrm{tr}\big[M\,|\phi\rangle\langle\phi|\big]$$

$\Rightarrow$ $\rho = |\phi\rangle\langle\phi|$ (projector onto $|\phi\rangle$).

<u>Partial trace:</u>

Given <u>general</u> state ρ_{AB} on $A+B$ (e.g. $\rho_{AB} = |\psi\rangle\langle\psi|$), what is descr. of meas M_A on A?

$$\mathrm{tr}\big[(M\otimes I)\,\rho_{AB}\big] = \sum_{ij\,i'j'} \langle ij|M\otimes I|i'j'\rangle\langle i'j'|\rho_{AB}|ij\rangle$$

$$\delta_{jj'}$$

$$= \sum \langle i|M|i'\rangle\langle i'j|\rho_{AB}|ij\rangle = \mathrm{tr}\big[M\cdot\rho_A\big],$$

Where we def. the <u>partial trace</u>

$$\rho_A = \sum |i'\rangle\langle i' j|\, \rho_{AB}\, |ij\rangle\langle i|$$

$$= \sum \left(\mathbb{1}_A \otimes \langle j'|_B\right)\left(\rho_{AB}\right)\left(\mathbb{1}_A \otimes |j\rangle_B\right)$$

$$= \sum_j \langle j|_B\, \rho_{AB}\, |j\rangle_B$$

$$=: \underline{\underline{\mathrm{tr}_B\, \rho_{AB}}}$$

(In components: $\left(\mathrm{tr}_B\, \rho_{AB}\right)_{ii'} = \sum_j \left(\rho_{AB}\right)_{(ij)(i'j)}$)

Is any density matrix ρ <u>physical</u>?

Take $\rho = \sum \lambda_i |\phi_i\rangle\langle\phi_i|$ eigenval. decomp., and

def $|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\, |\phi_i\rangle_A |i\rangle_B$ ("purification" of ρ)

$$\Rightarrow \mathrm{tr}_B\left[|\psi\rangle_{AB}\langle\psi|_{AB}\right] = \mathrm{tr}_B\left[\sum \sqrt{\lambda_i \lambda_j}\, |\phi_i\rangle\langle\phi_j| \otimes |i\rangle\langle j|\right]$$

$$= \sum \lambda_i |\phi_i\rangle\langle\phi_i| = \rho \quad \Rightarrow \underline{\underline{\text{yes}}} \checkmark$$

Density matrix can serve as <u>alternative definition</u> of state.

Ensemble interpretation of density matrix

Consider $|\psi\rangle_{AB} = \alpha|00\rangle + \beta|11\rangle$

$$\Rightarrow \rho_A = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} = |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 \, |1\rangle\langle 1|$$

$$\mathrm{tr}[\Pi \rho_A] = |\alpha|^2 \, \mathrm{tr}[\Pi |0\rangle\langle 0|] + |\beta|^2 \, \mathrm{tr}[\Pi |1\rangle\langle 1|]$$

$\Rightarrow$ Can be interpreted as having $|0\rangle$ w/ prob. $p_0 = |\alpha|^2$ & $|1\rangle$ w/ prob. $p_1 = |\beta|^2$. "<u>ensemble interpretation</u>"

$\Rightarrow$ Is this consistent w/ <u>pure</u> state $|\psi\rangle_{AB}$?

Let B do proj. meas. in Z basis:

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle \quad \begin{array}{l} \xrightarrow{p_0 = |\alpha|^2} |\psi_0\rangle_A = |0\rangle_A \\ \text{Z meas. on B} \\ \xrightarrow{p_1 = |\beta|^2} |\psi_1\rangle_A = |1\rangle_A \end{array}$$

$\Rightarrow$ Alice doesn't know outcome $\Rightarrow$ <u>ensemble</u>

$$\{(p_0; |0\rangle), (p_1; |1\rangle)\} \equiv \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}.$$

(<u>Note</u>: Bob's description is different: he knows outcome and would describe the state as <u>either</u> $|0\rangle\langle 0|$ <u>or</u> $|1\rangle\langle 1|$)

But: Bob could also meas. in $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ basis!

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle \xrightarrow{X\text{ meas.}} \quad p_+ = \frac{|\alpha|^2+|\beta|^2}{2} = \frac{1}{2} \qquad |\psi_+\rangle_A = \frac{\alpha|0\rangle + \beta|1\rangle}{|\alpha|^2+|\beta|^2}$$

$$p_- = \frac{|\alpha|^2+|\beta|^2}{2} = \frac{1}{2} \qquad |\psi_-\rangle_A = \frac{\alpha|0\rangle - \beta|1\rangle}{|\alpha|^2+|\beta|^2}$$

not orthogonal!

Different ensemble $\quad \rho_A = p_+|\psi_+\rangle\langle\psi_+| + p_-|\psi_-\rangle\langle\psi_-|$

for same state $\Longrightarrow$ ens. interpretation ambiguous!

(Number of terms can vary ($\rightarrow$ HW!), non-orth. states as $|\psi_\pm\rangle$, ...)

How are diff. ensembles related?

Note: Not orthogonal, just $\langle\psi_i|\psi_i\rangle = 1$

Theorem: Let $\rho = \sum p_i |\psi_i\rangle\langle\psi_i| = \sum q_j |\phi_j\rangle\langle\phi_j|$.

Then, there exists a unitary $U = (u_{ij})$ s.th.

$$\sqrt{p_i}\,|\psi_i\rangle = \sum_j u_{ij}\sqrt{q_j}\,|\phi_j\rangle,$$

and vice versa. (If there are less j's than i's, pad with zeros, and vice versa.)

Proof: "$\Leftarrow$": let $\sqrt{p_i}\,|\psi_i\rangle = \sum_j u_{ij}\sqrt{q_j}\,|\phi_j\rangle$.

Then
$$\sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i \Big(\sum_j u_{ij}\sqrt{q_j}\,|\phi_j\rangle\Big)\Big(\sum_{j'} u^*_{ij'}\sqrt{q_{j'}}\,\langle\phi_{j'}|\Big)$$
$$= \sum_{jj'} \sqrt{q_j q_{j'}}\,|\phi_j\rangle\langle\phi_{j'}| \underbrace{\Big(\sum_i u^*_{ij'} u_{ij}\Big)}_{=\delta_{jj'}}$$
$$= \sum q_j |\phi_j\rangle\langle\phi_j| .$$

"$\Rightarrow$": Homework / see later (equiv. of purifications).

4. Schmidt decomposition and purifications

Given $|\psi\rangle_{AB}$ bipartite, let

$$\mathrm{tr}_B |\psi\rangle\langle\psi| = \rho_A = \sum p_i |i\rangle_A \langle i|_A$$

with $|i\rangle_A$ eigenbasis (ONB) ("abuse" of notation...)

Choose <u>some</u> ONB $|a_j\rangle_B$ of B, expand

$$|\psi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle_A |a_j\rangle_B$$
$$= \sum_i |i\rangle_A \Big(\sum_j c_{ij} |a_j\rangle_B\Big) \quad =: |b_i\rangle : \ \underline{\text{no ONB}}$$

$$\ldots = \sum |i\rangle_A |b_i\rangle_B$$

We have $\sum_i p_i |i\rangle\langle i| = \mathrm{tr}_B \, |\psi\rangle\langle\psi| = \mathrm{tr}_B \Big(\sum_{ii'} |i\rangle\langle i'|_A \otimes |b_i\rangle\langle b_{i'}|_B \Big)$

$$= \sum_{ii'} |i\rangle\langle i'|_A \otimes \mathrm{tr}\big(|b_i\rangle\langle b_{i'}|_B\big)$$

$$= \sum_{ii'} \langle b_{i'} | b_i \rangle \cdot |i\rangle\langle i'|$$

Since $|i\rangle\langle i'|$ is basis (lin. indep.) in space of matrices:

$$\Longrightarrow \quad \langle b_{i'} | b_i \rangle = \delta_{i'i}\, p_i$$

$$\Longrightarrow \quad |i\rangle_B := \frac{1}{\sqrt{p_i}} |b_i\rangle \quad \text{is } \underline{\text{ONB for } B}$$

<u>different</u> basis than $|i\rangle_A$ ($\rightarrow$ notation!)

<u>Schmidt decomposition</u>:

Any $|\psi\rangle_{AB}$ can be written as

$$|\psi\rangle_{AB} = \sum_i \lambda_i \, |i\rangle_A |i\rangle_B \,,$$

with ONBs $|i\rangle_A$ & $|i\rangle_B$. The $\lambda_i = \sqrt{p_i} \geq 0$ are called <u>Schmidt coefficients</u>.

Note: $\rho_B = tr_A |\psi\rangle\langle\psi| = \sum_i \lambda_i^2 |i\rangle_B \langle i|_B$

$\Rightarrow |i\rangle_B$ eigenbasis of ρ_B!

$\Rightarrow$ If p_i non-degen.: Schmidt decomp. obtained by pairing eigenvectors of ρ_A & ρ_B.

Important consequence: For pure states $|\psi\rangle_{AB}$, ρ_A and ρ_B have the same eigenvalues!

How is Schmidt dec. related to other expansions?

$$|\psi\rangle = \sum c_{ij} |x_i\rangle_A |y_j\rangle_B$$

some ONBs

$$= \sum \lambda_k |k\rangle_A |k\rangle_B .$$

$|x_i\rangle_A, |y_j\rangle_B, |k\rangle_A, |k\rangle_B$ ONBs

$\Rightarrow \exists$ unitaries u_{ik}, v_{jk} s.th.

$$|k\rangle_A = \sum u_{ik} |x_i\rangle_A ; \quad |k\rangle_B = \sum v_{jk}^* |y_j\rangle_B$$

(pad w/ zeros if necessary...)

$$\Rightarrow \sum c_{ij} \, |x_i\rangle_A |y_j\rangle_B = \sum \lambda_k u_{ik} v_{jk}^* \, |x_i\rangle_A |y_j\rangle_B$$

$\xRightarrow{\text{lin. indep. of } |x_i\rangle_A |y_j\rangle_B}$
$$c_{ij} = \sum_k u_{ik} \lambda_k v_{jk}^* ,$$

$$\text{or} \quad C = U \cdot D \cdot V^\dagger \qquad \left(C \equiv (c_{ij}) \right)$$

with U, V unitary, and $D = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & 0 \\ 0 & & \lambda_k & \\ & & & 0 \end{pmatrix}$

$\boxed{\text{"Singular value decomposition" (SVD) of } C}$

(Derivation of SVD ($\to$ HW): U diagonalizes $CC^\dagger$, V $C^\dagger C$
$\Leftrightarrow$ derivation of Schmidt decomp.!)

Remark: Any two states $|\phi\rangle, |\psi\rangle$ w/ ident. Schmidt coeffs are related by local unitaries, i.e.

$$\exists\, U, V : \; |\phi\rangle = U \otimes V |\psi\rangle .$$

$\Rightarrow$ the λ_i contain all non-local properties,
$\lambda_1 \geq \lambda_2 \geq \ldots$

Proof: $|\phi\rangle = \sum \lambda_i |\phi_i^A\rangle \otimes |\phi_i^B\rangle$ (ONBs)

$|\psi\rangle = \sum \lambda_i |\psi_i^A\rangle \otimes |\psi_i^A\rangle$ (ONBs)

$|\phi_i^A\rangle, |\psi_i^A\rangle$ ONB $\Rightarrow \exists U : |\phi_i^A\rangle = U|\psi_i^A\rangle \ \forall i$

& same for B : $\exists V : |\phi_i^B\rangle = V|\psi_i^B\rangle \ \forall i$ $\square$

(Again: Pad w/ 0 if necessary.)

<u>Purification:</u>

Any $|\psi\rangle_{AB}$ s.th. $\mathrm{tr}_B |\psi\rangle\langle\psi|_{AB} = \rho_A$ is called a <u>purification</u> of ρ_A.

need <u>not</u> be orthogonal!

(E.g. $\rho_A = \sum p_i |\psi_i\rangle\langle\psi_i| \Rightarrow \sum \sqrt{p_i} |\psi_i\rangle|i\rangle$ is purif.)

Given two purifications $|\phi\rangle$ & $|\psi\rangle$ of ρ_A, what is their relation?

Write $|\phi\rangle, |\psi\rangle$ in <u>Schmidt form</u>:

$|\phi\rangle = \sum \lambda_i |\phi_i^A\rangle |\phi_i^B\rangle$ (all ONBs)

$|\psi\rangle = \sum \mu_i |\psi_i^A\rangle |\psi_i^B\rangle$

λ_i, μ_i wlog descending.

We have

$$\sum \lambda_i |\phi_i^A\rangle\langle\phi_i^A| = \mathrm{tr}_B |\phi\rangle\langle\phi| = \mathrm{tr}_B |\psi\rangle\langle\psi| = \sum \mu_i |\psi_i^A\rangle\langle\psi_i^A|$$

$\Rightarrow \lambda_i = \mu_i,\ |\phi_i^A\rangle = |\psi_i^A\rangle$ (up to phase)

if λ_i non-degen. (degen. $\to$ HW)

Now choose U s.th. $U|\phi_i^B\rangle = |\psi_i^B\rangle\ \forall i$ ($\Rightarrow U$ unitary)

$\Rightarrow\ |\psi\rangle = (\mathbb{1}\otimes U)|\phi\rangle.$

> All purifications are related by a unitary on the purifying system.

(Note: Closely related to unitary equivalence of ensemble decompositions $\to$ HW!)

36. Mixed states – unitary evolution + projective measurement

Unitary evolution of mixed state

How does a mixed state ρ_A evolve under a unitary U_A?

Consider purification $|\psi\rangle_{AB}$, $\mathrm{tr}_B|\psi\rangle\langle\psi| = \rho_A$.

$$|\psi\rangle \longmapsto (U_A\otimes \mathbb{1}_B)|\psi\rangle$$

$$\Rightarrow \rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi| \longmapsto \mathrm{tr}_B\left[(U_A \otimes \mathbb{1}_B)|\psi\rangle\langle\psi|(U_A^\dagger \otimes \mathbb{1}_B)\right]$$

$$= U_A \cdot \mathrm{tr}_B\left[(\mathbb{1}_A \otimes \mathbb{1}_B)|\psi\rangle\langle\psi|(\mathbb{1}_A \otimes \mathbb{1}_B)\right] U_A^\dagger$$

$$= \underline{\underline{U_A \rho_A U_A^\dagger}}$$

(Alt. derivation: $\rho_A = \sum p_i |\psi_i\rangle\langle\psi_i|$ & $|\psi_i\rangle \mapsto U_A|\psi_i\rangle$)

<u>Measurement of mixed states:</u>

Proj. measurement E_n:

Have seen: $p_n = \mathrm{tr}[E_n \rho_A]$.

Post-meas. state:

$$\rho_{A,n} = \frac{1}{p_n} \mathrm{tr}_B\left[(E_n \otimes \mathbb{1})|\psi\rangle\langle\psi|(E_n^\dagger \otimes \mathbb{1})\right]$$

$$= E_n \rho_A E_n^\dagger .$$