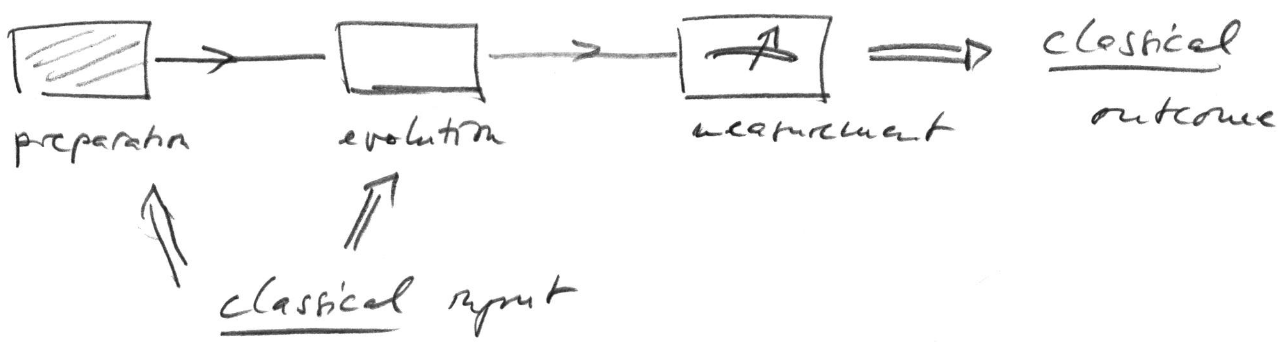
II. The formalism: States, measurements, evolution

1. Pure states, unitary evolution, projective measurements

Q.M. setup:



Q.M. system $\longrightarrow$ Hilbert space $\mathcal{H} \cong \mathbb{C}^d$

(Q.Info: typ. finite dim. H.S.)

State of system: vector $|\psi\rangle \in \mathcal{H}$ with $\||\psi\rangle\|^2 = \langle\psi|\psi\rangle = 1.$

(more precisely: rays $|\psi\rangle \sim e^{i\phi}|\psi\rangle$)

Use ket-bra notation:

$|v\rangle \in \mathbb{C}^d$: column vector

$\langle v| = (|v\rangle)^\dagger$: row vector

$\langle w|v\rangle$: scalar product $(\vec{w}^\dagger \vec{v} = \vec{w}^* \cdot \vec{v})$

Basis notation:

"Computational basis" $|0\rangle, |1\rangle, \dots |d-1\rangle$ of $\mathbb{C}^d$

$$|k\rangle \mathrel{\hat{=}} e_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k\text{'th position}$$

$$|v\rangle = \sum_{k=0}^{d-1} v_k |k\rangle = \begin{pmatrix} v_0 \\ \vdots \\ v_{d-1} \end{pmatrix}$$

Linear operators:

$M : \mathbb{C}^d \to \mathbb{C}^d$ is linear:

$$M\left(\alpha|v\rangle + \beta|w\rangle\right) = \alpha M(|v\rangle) + \beta M(|w\rangle)$$

Write $M|v\rangle \equiv M(|v\rangle)$.

Matrix notation / expansion:

$$M = \left(\sum_{i=0}^{d-1} |i\rangle\langle i|\right) M \left(\sum_{j=0}^{d-1} |j\rangle\langle j|\right) =$$

$$= \sum_{i,j=0}^{d-1} M_{ij} |i\rangle\langle j| = \begin{pmatrix} M_{00} & M_{01} & \cdots & \\ M_{10} & \ddots & & \\ \vdots & & \ddots & \\ & & & M_{d-1,d-1} \end{pmatrix}$$

with $M_{ij} = \langle i|M|j\rangle$.

(i) Preparation:

Prepares known initial state $|\phi\rangle \in \mathbb{C}^d$.

(ii) Evolution:

Evolution = unitary transformation $U: \mathbb{C}^d \to \mathbb{C}^d$;

$$|\phi\rangle \longmapsto U|\phi\rangle.$$

$$U \text{ unitary} :\Leftrightarrow U^\dagger U = UU^\dagger = \mathbb{1} \leftarrow \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\left(\text{or: } \sum (U^\dagger)_{ij} U_{jk} = \sum \bar{U}_{ji} U_{jk} = \delta_{ik}\right)$$

(Note 1: $\langle\phi|U^\dagger U|\phi\rangle = \langle\phi|\phi\rangle = 1 \Rightarrow$ norm preserved, if and only if U unitary.)

(Note 2: U can in part. be generated by time evolution w/ Hamiltonian.)

(iii) Measurement:

Observable quantities $\equiv$ hermitian operator $A = A^\dagger$

Eigenvalue decomposition:

$$A = \sum_n a_n E_n \; ; \quad E_n^2 = E_n = E_n^\dagger \text{ projector onto}$$

eigenspace (e.g. $E_n = |\psi_n\rangle\langle\psi_n|$)

Measurement of A in state $|\phi\rangle$:

Outcome a_n w/ prob. $p_n = \langle\phi|E_n|\phi\rangle = \| E_n|\phi\rangle \|^2$

$$\left(= |\langle \psi_n|\phi\rangle|^2 \right)$$

$$\left(\text{Note: } \sum p_n = \langle\phi| \underbrace{\sum E_n}_{=\mathbb{1}} |\phi\rangle = \langle\phi|\phi\rangle = 1\right)$$

State after meas.:

$$|\phi_n\rangle = \frac{E_n|\phi\rangle}{\| E_n|\phi\rangle \|}$$

Expectation value:

$$\langle\phi|A|\phi\rangle = \sum a_n \langle\phi|E_n|\phi\rangle$$

2. Composite systems

Consider system w/ two <u>separate parts</u> ("subsystems")
A (= Alice) and B (= Bob).

A B

Hilbert space   Hilbert space

$\mathcal{H}_A$ $\mathcal{H}_B$

$\Rightarrow$ Joint system: Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

What is general form of $|\phi\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$?

$|i\rangle_A$ basis of $\mathcal{H}_A = \mathbb{C}^{d_A}$

$|j\rangle_B$ basis of $\mathcal{H}_B = \mathbb{C}^{d_B}$

$\Rightarrow$ $|i\rangle_A \otimes |j\rangle_B \equiv |i\rangle_A |j\rangle_B \equiv |i,j\rangle_{AB} \equiv |ij\rangle_{AB} \equiv |ij\rangle$

is a basis of $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \cong \mathbb{C}^{d_A d_B}$,

$i = 0, \dots, d_A - 1$; $j = 0, \dots, d_B - 1$.

General state $|\phi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$|\phi\rangle_{AB} = \sum_{\substack{i=0,\dots,d_A-1 \\ j=0,\dots,d_B-1}} c_{ij} |i\rangle_A |j\rangle_B \ : \ d_A \cdot d_B\text{-dim. vector } (c_{ij})_{ij}$$

What if A acts w/ M_A on her system, and/or B w/ N_B on his?

(Note: M_A, N_B could be unitaries, measurements (E_n), or "doing nothing", $N_B = \mathbb{1}_B$.)

Consider first $|\phi\rangle_{AB} = |i\rangle_A \otimes |j\rangle_B$. (14)

Action of A should only change her system (as if B wasn't there):

$$|i\rangle_A \longmapsto M_A |i\rangle_A \quad , \quad |i\rangle_A \otimes |j\rangle_B \longmapsto (M_A |i\rangle_A) \otimes |j\rangle_B$$

Same for Bob, jointly:

$$|i\rangle_A \otimes |j\rangle_B \longmapsto M_A |i\rangle_A \otimes N_B |j\rangle_B$$

$$\equiv (M_A \otimes N_B)\, |i\rangle_A \otimes |j\rangle_B$$

<u>Linearity</u>:

$$|\phi\rangle_{AB} \longmapsto (M_A \otimes N_B)\, |\phi\rangle_{AB}$$

Matrix elements:

$$\langle i_A i_B | M_A \otimes M_B | j_A j_B \rangle = \langle i_A | M_A | j_A \rangle \langle i_B | M_B | j_B \rangle$$

$$\| \qquad\qquad\qquad \|$$

$$(M_A \otimes M_B)_{(i_A i_B),(j_A j_B)} \qquad (M_A)_{i_A, j_A} \cdot (N_B)_{i_B, j_B}$$

$$M_A \otimes N_B = \left(\begin{array}{c|c|c} (M_A)_{00} \cdot N_B & (M_A)_{10} \cdot N_B & \cdots \\ \hline (M_A)_{01} \cdot N_B & \cdots & \\ \hline \vdots & & \end{array}\right)$$

Examples:

Qubit: $\mathcal{H} = \mathbb{C}^2$,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle ; \qquad |\alpha|^2 + |\beta|^2 = 1$$

Observable $$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underbrace{|0\rangle\langle 0|}_{\substack{E_0 \\ a_0 = +1}} - \underbrace{|1\rangle\langle 1|}_{\substack{E_1 \\ a_1 = -1}}$$

Measurement:

outcome $a_0 = +1$ w/ prob. $\langle\psi|E_0|\psi\rangle = |\alpha|^2$

outcome $a_1 = -1$ w/ prob. $\langle\psi|E_1|\psi\rangle = |\beta|^2$

Observable $$X = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underbrace{|+\rangle\langle +|}_{E_+} - \underbrace{|-\rangle\langle -|}_{E_-}$$

with $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

Measurement: outcomes $\pm$ w/ prob. $\left|\langle\pm|(\alpha|0\rangle + \beta|1\rangle)\right|^2 = \frac{|\alpha \pm \beta|^2}{2}$

Evolution: $U = H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ "Hadamard gate"

$$U|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}(\alpha|0\rangle + \beta|1\rangle)$$

$$= \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

Meas. in Z-basis $\{|0\rangle, |1\rangle\}$

outcome 0 w/ prob. $\frac{|\alpha+\beta|^2}{2}$

outcome 1 w/ prob. $\frac{|\alpha-\beta|^2}{2}$

H transforms betw. X and Z basis.

In fact, $H = |+\rangle\langle 0| + |-\rangle\langle 1| = |0\rangle\langle +| + |1\rangle\langle -| = H^T$

Measurement on bipartite state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$

Alice & Bob measure Z:

project onto $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

$\Rightarrow \quad P_{01} = P_{10} = \frac{1}{2}; \ P_{00} = P_{11} = 0$

Alice & Bob measure X:

project onto $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$:

$|\langle ++|\psi\rangle|^2 = 0$

$|\langle +-|\psi\rangle|^2 = \left|-\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{2}$

$|\langle -+|\psi\rangle|^2 = \ldots = \frac{1}{2}$

$|\langle --|\psi\rangle|^2 = 0$

(using $\langle +|0\rangle = \langle +|1\rangle = \langle -|0\rangle = -\langle -|1\rangle = \frac{1}{\sqrt{2}}$)

$\Rightarrow$ perfect anti-correlation

(in fact, in any basis $\to$ Homework!)

Alice meas X, Bob meas Z:

$$|\langle +0|\psi\rangle|^2 = \left|-\tfrac{1}{2}\right|^2 = \tfrac{1}{4}$$

$$|\langle +1|\psi\rangle|^2 = \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{4}$$

$$|\langle -0|\psi\rangle|^2 = \quad \ldots \quad = \tfrac{1}{4}$$

$$|\langle -1|\psi\rangle|^2 = \quad \ldots \quad = \tfrac{1}{4}$$

Outcomes for A & B are separately completely random, but <u>outcomes in the same basis</u> are perfectly anti-corr.

3. Mixed States

Consider bipart. state $|\psi\rangle_{AB} = \sum c_{ij} |i\rangle|j\rangle$

We have only access to A.

→ How can we characterize measurement on A?

Meas. M on A $\Longleftrightarrow$ meas. $M_A \otimes \mathbb{1}_B$ on A+B.

$$\langle\psi| M_A \otimes \mathbb{1}_B |\psi\rangle = \sum c_{i'j'}^* \langle i'|\langle j'| (M_A \otimes \mathbb{1}_B) |i\rangle|j\rangle c_{ij}$$

$$= \sum c_{i'j'}^* c_{ij} \langle i'|M_A|i\rangle \underbrace{\langle j'|j\rangle}_{=\delta_{jj'}}$$

$$= \sum_{ii'} \left(\sum_j c_{i'j}^* c_{ij} \right) \langle i'|M_A|i\rangle = (*)$$

Define ρ_A ($d_A \times d_A$ matrix) via $(\rho_A)_{ii'} = \sum_j c_{i'j}^* c_{ij} = \tilde{C}\tilde{C}^\dagger$

(with $\tilde{C} = (c_{ij})_{ij}$), or equiv. $\rho_A = \sum_{ii'j} c_{i'j}^* c_{ij} |i\rangle\langle i'|$

... $(*) = \mathrm{tr}[\rho_A M]$,

with the <u>trace</u> $\mathrm{tr}(X) = \sum \langle k|X|k\rangle$. ONB!

<u>Note</u>: The trace is cyclic: $\mathrm{tr}(AB) = \sum_k \langle k|AB|k\rangle$

$$= \sum_{kl} \langle k|A|l\rangle\langle l|B|k\rangle = \sum_{kl} \langle l|B|k\rangle\langle k|A|l\rangle$$

$$= \sum \langle l|BA|l\rangle = \mathrm{tr}(BA),$$

and thus basis-indep: $\mathrm{tr}(X) = \mathrm{tr}(U^\dagger U X) = \mathrm{tr}(U X U^\dagger)$. (19)

ρ_A is called <u>density operator</u> or <u>density matrix</u>, or <u>mixed state</u>.
It characterizes systems where we only have <u>partial knowledge</u>.

<u>Properties of ρ_A:</u>

- $\rho_A = \tilde{C}\tilde{C}^\dagger \Rightarrow \rho_A^\dagger = (\tilde{C}\tilde{C}^\dagger)^\dagger = \tilde{C}\tilde{C}^\dagger = \rho_A$
- ρ_A is positive semi-definite ($=$ all eigenvalues ≥ 0), $\rho_A \geq 0$:

$$\langle\phi|\rho_A|\phi\rangle = \langle\phi|\tilde{C}\tilde{C}^\dagger|\phi\rangle = (\tilde{C}^\dagger|\phi\rangle)^\dagger(\tilde{C}^\dagger|\phi\rangle) \geq 0 \quad \forall|\phi\rangle.$$

- $\mathrm{tr}(\rho_A) = \sum_i (\tilde{C}\tilde{C}^\dagger)_{ii} = \sum_{ij} c_{ij} c_{ij}^* = \langle\psi|\psi\rangle = 1.$

<u>Properties of density operators:</u>

- $\rho_A^\dagger = \rho_A$
- $\rho_A \geq 0$
- $\mathrm{tr}(\rho_A) = 1$

<u>Note:</u> Consequence: For $0 < p < 1$, ρ, ρ' density ops, $p\rho + (1-p)\rho'$ is also density op $\Rightarrow$ density ops form <u>convex set</u>!

Is ρ_A uniquely determined by

$$\mathrm{tr}[M\rho_A] = \langle\psi|_{AB}\ M\otimes\mathbb{1}\ |\psi\rangle_{AB}\ ?$$

Yes: $\mathrm{tr}[X^\dagger Y]$ is scalar product, & overlap of ρ_A w/ all herm. M determines herm. part of ρ_A entirely!

(Consequence: All numbers in ρ_A meaningful $\Rightarrow$ no phase ambiguity!)

What is ρ_A for pure state $|\phi\rangle_A$?

$$\langle\phi|M|\phi\rangle = \mathrm{tr}[\langle\phi|M|\phi\rangle] = \mathrm{tr}[M\,|\phi\rangle\langle\phi|]$$

$\Rightarrow\ \rho = |\phi\rangle\langle\phi|$ (projector onto $|\phi\rangle$).

Partial trace:

Given general state ρ_{AB} on $A+B$ (e.g. $\rho_{AB} = |\psi\rangle\langle\psi|$), what is descr. of meas M_A on A?

$$\mathrm{tr}[(M\otimes I)\,\rho_{AB}] = \sum_{ij\,i'j'} \langle ij|M\otimes I|i'j'\rangle\langle i'j'|\rho_{AB}|ij\rangle$$

$$\delta_{jj'}$$

$$= \sum \langle i|M|i'\rangle\langle i'j|\rho_{AB}|ij\rangle = \mathrm{tr}[M\cdot\rho_A],$$