Entanglement in many-body states: The area law

Up to now: Locality

$\longrightarrow$ light cone (LR-bnd.)

$\longrightarrow$ exp. clustering of correlations:
Correlations local

$\longrightarrow$ correlations stable w/in phase (= gapped path)

Question: Do local correlations imply state is simple (e.g. close to product, or has simple class. description)?

$\rightarrow$ Not necessarily. Depends on correlations measured!

$\rightarrow$ e.g. for a maximally ent. state $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$, meas. of one part gives no information!

$\rightarrow$ random quantum states on N spins: measurement on any "small" ($< N/2$) part looks completely random ($\rightarrow$ uncorrelated), but state is very complex
[$\rightarrow$ 'Levy's lemma', concentration of measure]

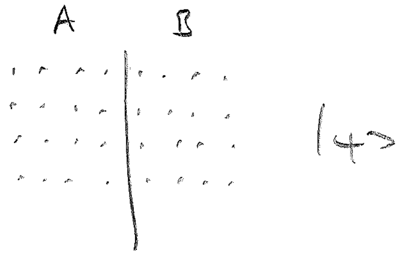
$\rightarrow$ Toric Code: no correlations for any local observables, but not product-state-like: Loop ops correlated!

How to assess "sum" of all type of quantum correlations in a state?

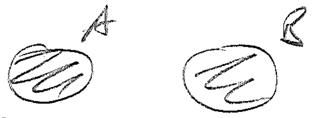
$\longrightarrow$ Entanglement.

<u>What is entanglement?</u>

<u>Bipartition:</u>



$|\psi\rangle =$



$\mathcal{H}_A \otimes \mathcal{H}_B$

- $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$: product state; no entanglement.

- $|\psi\rangle = \alpha_1 |\psi_A^1\rangle \otimes |\psi_B^1\rangle + \alpha_2 |\psi_A^2\rangle \otimes |\psi_B^2\rangle + \ldots$

  $\longrightarrow$ correlations betw. A & B $\Rightarrow$ entanglement.

  "Amount" should dep. on $|\alpha_i|^2$ and overlaps $\langle \psi_A^i | \psi_A^j \rangle$, $\langle \psi_B^i | \psi_B^j \rangle$.

- How can we quantify entanglement in a meaningful way?

Interlude: The singular value decomposition (SVD).

SVD: Any matrix M can be written as

$$M = U D V^\dagger \,, \quad \text{with } U, V \text{ unitary, and}$$

$$D = \underbrace{\left(\begin{array}{cccc|c} s_1 & & & 0 & \\ & s_2 & & & 0 \\ & & \ddots & & \\ 0 & & & s_m & \end{array} \right)}_{n} \Big\} \, m \quad \text{diagonal.}$$

$s_1 \geq s_2 \geq \dots \geq s_m$: "singular values".

Note: U, V unique up to degenerate s_i's.

Proof/Computation of SVD:

Note that $M M^\dagger = U D V^\dagger V D^\dagger U^\dagger = U D D^\dagger U^\dagger$

has eigenvalues $\{ s_i^2 \}$.

$\to$ Diagonalize $M M^\dagger$: $M M^\dagger = U \Lambda U^\dagger$, U unitary.

Λ diag., positive $\Rightarrow s_i = \sqrt{\mathrm{eig}(\Lambda)} \Rightarrow D D^\dagger = \Lambda$.

For simplicity: M square, all $s_i \neq 0$ (general case: $\to$ Homework)

$\to$ Let $V^\dagger = D^{-1} U^\dagger M$. Then $U D V^\dagger = M$, and

$$V^\dagger V = D^{-1} \underbrace{U^\dagger M M^\dagger U}_{= \Lambda} D^{-1} = D^{-1} \Lambda D^{-1} = \mathbb{1} .$$

Back to entanglement:

Consider general bipartite state

$$|\psi\rangle = \sum_{ij} c_{ij} |\phi_A^i\rangle \otimes |\phi_B^j\rangle \quad \text{with ONB } |\phi_A^i\rangle, |\phi_B^j\rangle.$$

Use SVD of $C \equiv (c_{ij})$:

$$c_{ij} = \sum_k u_{ik} s_k \bar{v}_{jk}$$

$$\Rightarrow |\psi\rangle = \sum_k s_k \underbrace{\left(\sum_i u_{ik} |\phi_A^i\rangle\right)}_{=: |\psi_A^k\rangle} \otimes \underbrace{\left(\sum_j \bar{v}_{jk} |\phi_B^j\rangle\right)}_{=: |\psi_B^k\rangle}$$

again ONB

$$= \underline{\underline{\sum_k s_k |\psi_A^k\rangle \otimes |\psi_B^k\rangle}}. \quad \text{"\underline{Schmidt decomposition}"}$$

(s_k: Schmidt coefficients)

→ Orthogonal $|\psi_A^k\rangle$ → maximizes correlations.

Amount of correlations dep. on:

→ # of k: more k → more correlations.

→ Distribution of s_k: Prob. for $k \propto |s_k|^2$ ⇒

⇒ flat distribution has more entanglement

Note:
Any two states with the same Schmidt coefficients are related by basis trafo on A & B (and vice versa).

How to quantify entanglement?

① zero for product state.

② Invariant under local unitaries $\rightarrow$ only dep. on $|s_k|^2$.

③ Additive: $E(|\psi\rangle) + E(|\phi\rangle) = E(|\psi\rangle \otimes |\phi\rangle)$

$\rightarrow$ Logarithmic in #k's for flat distr. $s_k \equiv \hat{s}$.

④ Measure "randomness" in s_k^2.

⑤ Normalization: $E\left(\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)\right) = 1$ (or $\log 2$).

$\Rightarrow$ Use entropy of $s_k^2 =: p_k$: $\left(|\psi\rangle = \sum_k s_k |\psi_A^k\rangle \otimes |\psi_B^k\rangle\right)$

$$E(|\psi\rangle) = -\sum_k p_k \log p_k .$$

either $\log_2$ or $\ln$.

Can also be expressed in terms of the reduced density matrix:

$$|\psi\rangle = \sum_k s_k |\psi_A^k\rangle |\psi_B^k\rangle$$

$$\Rightarrow \quad \rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi| = \sum_m \underbrace{\langle\psi_B^m|\psi\rangle}_{= s_m |\psi_A^m\rangle}\langle\psi|\psi_B^m\rangle = \ldots$$

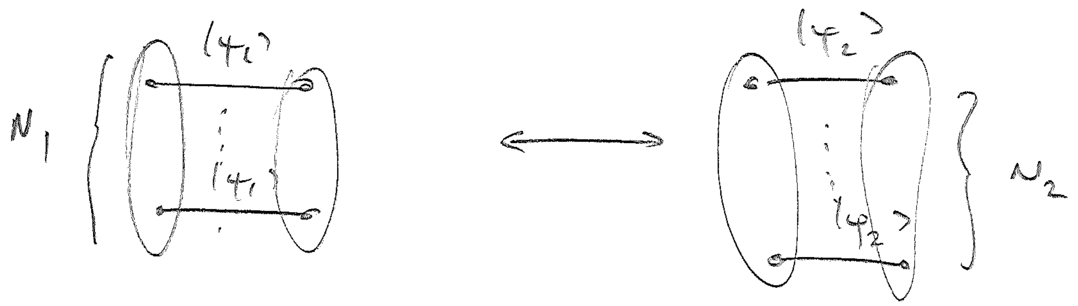
$$= \sum_m |s_m|^2 \, |\psi_A^m\rangle\langle\psi_A^m|$$

$\Rightarrow$ $p_k = s_k^2$ are eigenvalues of the reduced density matrix of $|\psi\rangle$ on A (or equiv. on B).

Define von Neumann entropy $S(\rho) = -\mathrm{tr}(\rho \log \rho)$.

$$\Rightarrow \quad E(|\psi\rangle) = S\left(\mathrm{tr}_B |\psi\rangle\langle\psi|\right) = S\left(\mathrm{tr}_A |\psi\rangle\langle\psi|\right).$$

Note: $E(|\psi\rangle)$ uniquely quantifies the "asymptotic entanglement content" of $|\psi\rangle$:



Reversible conversion by local ops. + class. communication if $N_1 E(|\psi_1\rangle) = N_2 E(|\psi_2\rangle)$ for $N_1, N_2 \to \infty$.

Alternative entropic measures possible:

Renyi' entropies $S_\alpha(\rho) = \frac{1}{1-\alpha} \log(\mathrm{tr}(\rho^\alpha)) \hat{=} \frac{\log(\sum p_k^\alpha)}{1-\alpha}$.

In particular: $\lim_{\alpha \to 1} S_\alpha(\rho) = S(\rho)$; $S_0(\rho) = \log \operatorname{rank} \rho$.

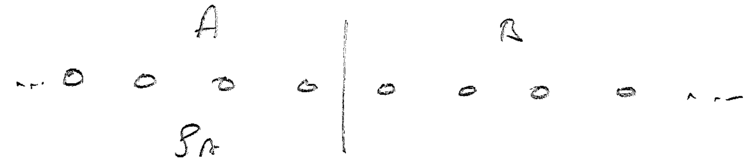
<u>Entanglement in many-body states:</u>

- Ground states of <u>local</u> <u>gapped</u> Hamiltonians.
- Entanglement "originates" from interactions.

  $\longrightarrow$ entanglement "local"?

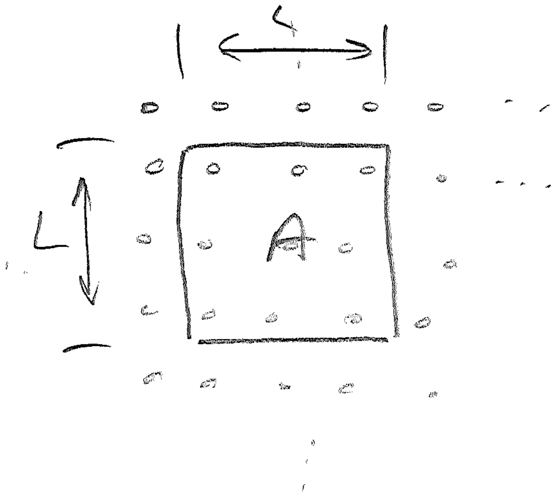
<u>Area law:</u> Entanglement of a region of a many-body ground state with complement (i.e., entropy of the red. state of the region) scales like the <u>boundary</u> of the region.

<u>1D</u>:



$S(\rho_A) \leq$ const. (indep. of size)

2D:



$$S(\rho_A) \le \underbrace{|\partial A|}_{\text{size of bnd.} \,\propto L}$$

etc. ...

→ entanglement in g. states very special!

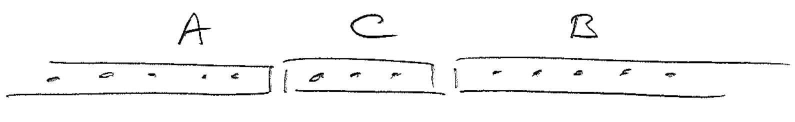
→ stronger than correlations: states with small corr. but lots of ent. exist

→ random states have extensive entropy!

Will see: Area law allows for succinct description of state!

How could we try to prove area law?

$\boxed{\text{Idea I}}$ exponential clustering?



$|\psi\rangle$

Corr(A:B) = 0 ⟹ ok:

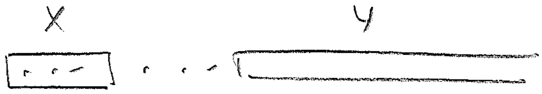
$$\Rightarrow |\psi\rangle = (\mathbb{1}_A \otimes U_C \otimes \mathbb{1}_B)(|\psi_{AC_1}\rangle \otimes |\psi_{C_2B}\rangle)$$

→ Ent. bounded by $\log|C|$.

<u>Problem 1</u>: $\mathrm{Corr}(A:B) \sim e^{-|C|/\xi}$, but factorization does not epsilonize!

(<u>Note</u>: Very recent result: If exp. clustering across <u>all</u> cuts, then area law follows: Brandao & Horodecki, Nat. Phys. '14)

<u>Problem 2</u>:



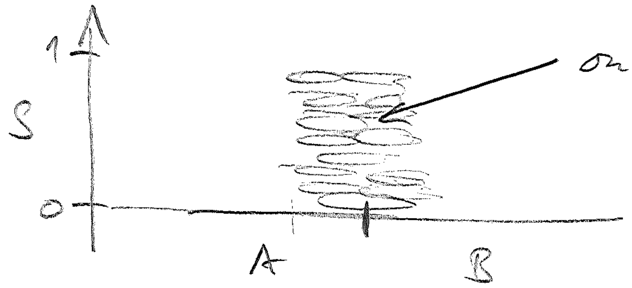
$\longrightarrow$ exp. clustering <u>scales with $|X|$</u> !!

Idea II: <u>Quasi-adiabatic continuation</u>

$$H_0 \xrightarrow{H_s} H_1 \quad \text{local, gapped path}$$

$\uparrow$
g.s. with area law.

$i\mathcal{D}_s$: quasi-local



only terms in $\mathcal{D}_s$ which cross cut create entanglement.
$\Rightarrow$ entanglement created at constant rate $\propto$ bond.
$s \in [0;1] \Rightarrow$ ent. grows prop. to <u>bond.</u>

Problem: Bound on entangling rate tricky.

Why? $\rightarrow$ ent. rate of terms at bnd. only: simple.

$\rightarrow$ But: other terms can "transport ent. away"

$\rightarrow$ Recently shown to still be true
(Van Acoleyen, Mariën, Verstraete, PRL'13)

$\Rightarrow$ area law stable within phase!

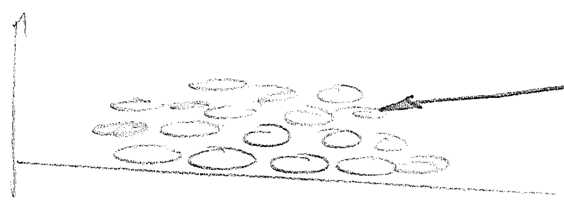
Without adiabatic path?

In 1D: Area law for any gapped Hamiltonian!

Proofs: • Hastings (JSTAT '07) using LR-bounds & filter functions: $|\psi\rangle\langle\psi| \approx (O_L \otimes O_R) \cdot O_{bnd}$

$$\boxed{S \sim e^{\underbrace{cv/\Delta}_{\propto \xi}}}$$

• Arad, Kitaev, Landau, Vazirani '13 based on ideas from computer science:



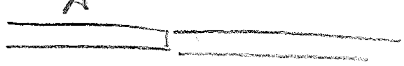
ops to "cool" system to G.S. $\rightarrow$ can select subset which has a "thin point".

$$\boxed{S \sim O\left(\frac{1}{\Delta}\right)} \longrightarrow \text{exp. improvement}$$

Summary of results on area laws:

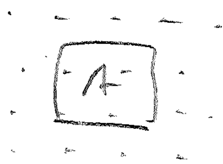
Area law: $S(\rho_A) \sim$ boundary of A

1D:



$S(\rho_A) \leq$ const.: proven ✓

2D:



$S(\rho_A) \leq c|\partial A|$

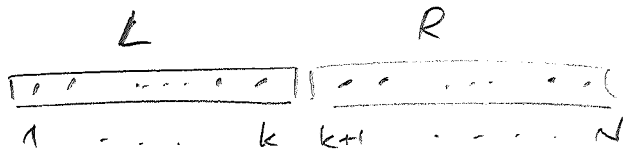
- stable within phase ✓
- holds for product states, Toric Code, ... ✓
- believed to be true for general gapped Ham.

Relevance of area law:

$\Rightarrow$ simple representation of QMB states (in 1D).

Simplest case:

Schmidt rank ($= \#$ of non-zero λ_i) is $\leq$ const for all cuts:



$$\forall k: \quad |\psi\rangle = \sum_{\alpha=1}^{D} \lambda_\alpha^k \, |l_\alpha^k\rangle \, |r_\alpha^k\rangle; \quad D \text{ const. (indep of } N).$$

Consider chain of N spins:

$$|\psi\rangle = c_{i_1 \dots i_N} |i_1, \dots, i_N\rangle$$

Interpret c as matrix w/ indices i_1 and $(i_2, \dots, i_N)$:

SVD:
$$c_{i_1 i_2 \dots i_N} = M_{i_1,(i_2 \dots i_N)} = \sum_{\alpha_1} \underbrace{U^{[1]}_{i_1 \alpha_1}}_{=: A^{[1]\alpha_1}_{i_1}} \underbrace{\lambda^{[1]}_{\alpha_1} V^{[1]}_{\alpha_1,(i_2, \dots, i_N)}}_{=: c^{[2]}_{\alpha_1 i_2 \dots i_N}}$$

with
$$\sum_{i_1} A^{[1]\alpha_1}_{i_1} \bar{A}^{[1]\beta_1}_{i_1} = \delta_{\alpha_1 \beta_1}$$

and
$$\sum_{i_2 \dots i_N} c^{[2]}_{\alpha_1, i_2 \dots i_N} \bar{c}^{[2]}_{\beta_1 i_2 \dots i_N} = |\lambda^{[1]}_{\alpha_1}|^2 \delta_{\alpha_1 \beta_1}$$

Now interpret $c^{[2]}$ as matrix w/ idx $(\alpha_1 i_2)$ and $(i_3 - i_N)$:

$$c^{[2]}_{(\alpha_1 i_2)(i_3 - i_N)} = \sum_{\alpha_2} \underbrace{U^{[2]}_{(\alpha_1 i_2)\alpha_2}}_{A^{[2]i_2}_{\alpha_1\alpha_2}} \underbrace{\lambda^{[2]}_{\alpha_2} V^{[2]}_{\alpha_2, (i_3 \ldots i_N)}}_{=: c^{[3]}_{\alpha_2 i_3 - i_N}}$$

$$\text{with} \quad \sum_{\alpha_1, i_2} A^{[2]i_2}_{\alpha_1\alpha_2} \bar{A}^{[2]i_2}_{\alpha_1\beta_2} = \delta_{\alpha_2\beta_2}$$

$$\text{and} \quad \sum_{i_3 - i_N} c^{[3]}_{\alpha_2 i_3 - i_N} c^{[3]}_{\beta_2 i_3 - i_N} = |\lambda^{[2]}_{\alpha_2}|^2 \delta_{\alpha_2\beta_2}$$

<u>etc., until position k:</u>

$$|\psi\rangle = \sum_{\substack{i_1 - i_N \\ \alpha_1 - \alpha_k}} A^{[1]i_1}_{\alpha_1} A^{[2]i_2}_{\alpha_1\alpha_2} \cdots A^{[k]i_k}_{\alpha_{k-1}\alpha_k} c^{[k+1]}_{\alpha_k i_{k+1} - i_N} |i_1, \ldots, i_k\rangle |i_{k+1}, \ldots, i_N\rangle$$

$$= \sum_{\alpha_k = 1}^{?} |\ell^k_{\alpha_k}\rangle |r^k_{\alpha_k}\rangle,$$

$$\text{with} \quad |\ell^k_{\alpha_k}\rangle = \sum_{\substack{i_1 - i_k \\ \alpha_1 - \alpha_{k-1}}} A^{[1]i_1}_{\alpha_1} A^{[2]i_2}_{\alpha_1\alpha_2} \cdots A^{[k]i_k}_{\alpha_{k-1}\alpha_k} |i_1, \ldots, i_k\rangle$$

$$\text{and} \quad |r^k_{\alpha_k}\rangle = \sum_{i_{k+1} - i_N} c^{[k+1]}_{\alpha_k i_{k+1} - i_N} |i_{k+1}, \ldots, i_N\rangle$$

Is this a <u>Schmidt decomposition</u>?

$$\underline{\langle r^k_{\alpha_k} | r^k_{\beta_k} \rangle} = \sum_{\substack{i_1 - i_N \\ j_1 - j_N}} c^{[k+1]}_{\alpha_k i_{k+1} - i_N} \bar{c}^{[k+1]}_{\beta_k j_{k+1} - j_N} \underbrace{\langle j_{k+1} - j_N | i_{k+1} - i_N \rangle}_{= \delta_{j_{k+1} i_{k+1}} \cdots \delta_{j_N i_N}}$$

$$= \underline{|\lambda^{[k]}_{\alpha_k}|^2 \delta_{\alpha_k \beta_k}} \Rightarrow \underline{\text{orthogonal}}$$

$$\Rightarrow \quad |\hat{r}^k_{\alpha_k}\rangle = \frac{1}{|\lambda^{[k]}_{\alpha_k}|} |r^k_{\alpha_k}\rangle \quad \text{is } \underline{\text{ONB}}.$$

$$\langle e^k_{\alpha_k} | e^k_{\beta_k} \rangle = \sum_{\substack{i_1 - i_k \\ \alpha_1 - \alpha_{k-1} \\ \beta_1 - \beta_{k-1}}} \underbrace{\underbrace{\underbrace{A^{[1] i_1}_{\alpha_1} \bar{A}^{[1] i_1}_{\beta_1}}_{= \delta_{\alpha_1 \beta_1}} \cdot A^{[2] i_2}_{\alpha_1 \alpha_2} \bar{A}^{[2] i_2}_{\beta_1 \beta_2}}_{\delta_{\alpha_2 \beta_2} \quad \cdots} \cdots A^{[k] i_k}_{\alpha_{k-1} \alpha_k} \bar{A}^{[k] i_k}_{\beta_{k-1} \beta_k}}_{\delta_{\alpha_k \beta_k}}$$

$\Rightarrow$ <u>ONB</u>

$\Rightarrow$ Schmidt decomposition

$$|\psi\rangle = \sum_{\alpha_k = 1}^{D} |\lambda^{[k]}_{\alpha_k}| \, |e^k_{\alpha_k}\rangle |\hat{r}^k_{\alpha_k}\rangle$$

$\Rightarrow$ rank / size of index α_k bounded by D for <u>all cuts</u> k.

Continue until last site:

$$c^{[N]}_{\alpha_{N-1} i_N} =: A^{[N] i_N}_{\alpha_{N-1}}$$

with $$\sum_{i_N} A^{[N] i_N}_{\alpha_{N-1}} A^{[N] i_N}_{\beta_{N-1}} = |\lambda^{[N]}_{\alpha_{N-1}}|^2 \, \delta_{\alpha_{N-1} \beta_{N-1}} .$$

$$\Rightarrow |\psi\rangle = \sum_{\substack{i_1 \dots i_N \\ \alpha_1 \dots \alpha_{N-1}}} \underbrace{A^{[1] i_1}_{\alpha_1} A^{[2] i_2}_{\alpha_1 \alpha_2} A^{[3] i_3}_{\alpha_2 \alpha_3} \cdots A^{[N-1] i_{N-1}}_{\alpha_{N-2} \alpha_{N-1}}}_{} A^{[N] i_N}_{\alpha_{N-1}} |i_1, \dots, i_N\rangle$$

$A^{[k] i_k}_{\beta \gamma}$ is <u>matrix</u> w indices β, γ $\Rightarrow$ <u>matrix product</u>!

(and $A^{[1] i_1}$, $A^{[N] i_N}$ row / col. vectors).

$$\Rightarrow \boxed{|\psi\rangle = \sum_{i_1 \dots i_N} A^{[1] i_1} \cdot A^{[2] i_2} \cdot \ldots \cdot A^{[N-1] i_{N-1}} \cdot A^{[N] i_N} |i_1, \dots, i_N\rangle}$$

$\boxed{\text{Matrix Product States}}$