## Applications of quasi-adiabatic evolution: Quantum phases

### Last lecture:

$$H_s = \sum_z h_{z,s} \quad \text{param.-dep. Hamiltonian}$$

### Quasi-adiabatic continuation:

$$i\mathcal{D}_s = \int F(\Delta t)\, e^{iH_s t} \left(\frac{d}{ds} H_s\right) e^{-iH_s t}\, dt$$

* $\frac{d}{ds}|\psi(s)\rangle = i\mathcal{D}_s\, |\psi(s)\rangle$ (unique G.S.)

* $\frac{d}{ds}|\psi_\alpha(s)\rangle = i\mathcal{D}_s\, |\psi_\alpha(s)\rangle + \underbrace{\sum Q_{\alpha\beta}|\psi_\beta(s)\rangle}_{\text{rotation within GS}}$ (approx. degen. G.S.)

* $H(s)$ local $\Rightarrow$ $\mathcal{D}_s$ approximately local

$\Rightarrow$ generates evolution along path by "time" $s$:

$$V(s) = \mathcal{T} \exp\left(\int_0^s ds'\, i\mathcal{D}_{s'}\right) \quad (\mathcal{T}: \text{"time"-ordered})$$

$\to$ $|\psi_\alpha(s)\rangle = \sum G_{\alpha\beta}\, V(s)\, |\psi_\beta(s)\rangle$

$\to$ $O(s) = V(s)^\dagger\, O\, V(s)$

$\Rightarrow$ $\langle\psi_\alpha(s)|O|\psi_{\alpha'}(s)\rangle = \sum \langle\psi_\beta(0)|\, \bar{G}_{\alpha\beta}\, O(s)\, G_{\alpha'\beta'}\, |\psi_{\beta'}(0)\rangle$

$D_s$ approx. local $\Rightarrow$ LR-type bound.

$\Rightarrow$ $O(s)$ is exp. close to a local operator (for fixed $s$):

$$\| O^\ell(s) - O(s) \| \sim e^{-\ell/\xi}$$

support on region of size $\ell$.

Application 1: Symmetry breaking

Transverse field Ising model:

$$H_s = \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z + s \sum_i \sigma_i^x$$

Symmetry $S := \prod \sigma_i^x$ : $[H_s, S] = 0$.

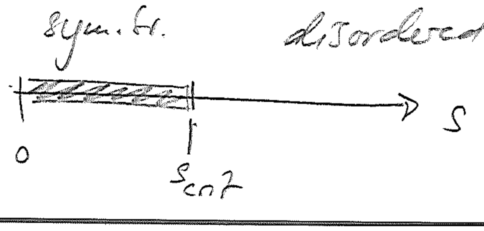
Limits: $s=0$: Ground states $|\psi_\uparrow\rangle = |\uparrow\ldots\uparrow\rangle$, $|\psi_\downarrow\rangle = |\downarrow\ldots\downarrow\rangle$.

$\rightarrow$ breaks symmetry $S$: $S|\psi_\uparrow\rangle = |\psi_\downarrow\rangle$.

$s \gg 1$: Ground state $|\psi_*\rangle = |+\ldots+\rangle$; $|+\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$

$\rightarrow$ no symmetry breaking, $S|\psi_x\rangle = |\psi_x\rangle$.

Phase diagram:



Are these different phases? Or could they be connected by a gapped path?

For $s=0$: Consider states $|\psi_\pm\rangle = \frac{|\psi_\uparrow\rangle \pm |\psi_\uparrow\rangle}{\sqrt{2}}$

$$\rightarrow \; \mathcal{S}|\psi_\pm\rangle = \pm|\psi_\pm\rangle$$

$$\rightarrow \; [H_s, \mathcal{S}] = 0 \Rightarrow \text{good quantum number}$$

$$\Rightarrow \forall s \quad |\psi_\pm(s)\rangle \text{ with } \mathcal{S}|\psi_\pm(s)\rangle = \pm|\psi_\pm(s)\rangle$$

eigenbasis.

$$\rightarrow \; \frac{d}{ds}H_s = \sum_i \sigma_i^x \Rightarrow \left[\frac{d}{ds}H_s, \mathcal{S}\right] = 0$$

$$\Rightarrow [i\mathcal{D}_s, \mathcal{S}] = 0 \Rightarrow G_{\alpha\beta} = \mathbb{1} \text{ in } |\psi_\pm\rangle\text{-basis.}$$

## Correlation functions:

$$1 = \langle\psi_\alpha|\sigma_i^z\sigma_j^z|\psi_\alpha\rangle = \langle\psi_\alpha(s)|\,\sigma_i^z(-s)\,\sigma_j^z(-s)\,|\psi_\alpha(s)\rangle \quad [\alpha=\pm]$$

and

$$\langle\psi_\alpha(s)|\,\sigma_i^z(-s)|\psi_\alpha(s)\rangle = \langle\psi_\alpha|\sigma_i^z|\psi_\alpha\rangle = 0.$$

$\Rightarrow$ connected correlation function $\langle O_i O_j\rangle_{\psi_\alpha} - \langle O_i\rangle\langle O_j\rangle = 1$,

with $O = \sigma_i^z(-s)$ approx. local.

In particular: Local $O^\ell$ with $\|O^\ell - O\| \le e^{-\ell/\xi}$:

$$|\langle O_i^\ell O_j^\ell \rangle - \langle O_i^\ell \rangle \langle O_j^\ell \rangle| \ge 1 - e^{-\ell/\xi} \quad [\text{with } \langle \cdot \rangle = \langle \psi_\pm | \cdot | \psi_\pm \rangle]$$

<u>On the other hand: $\xi \gg 1$</u>

$$|\psi_x\rangle = |+\rangle \dots |+\rangle$$

$\Rightarrow \langle \psi_x | \sigma_i^z | \psi_x \rangle = 0$ and $\langle \psi_x | \sigma_i^z \sigma_j^z | \psi_x \rangle = 0$

$$\Rightarrow |\langle O_i^\ell O_j^\ell \rangle - \langle O_i^\ell \rangle \langle O_j^\ell \rangle| \le e^{-\ell/\xi}.$$

In fact, for any local operator $Q$,

$$\langle \psi_x | Q_i Q_j | \psi_x \rangle = \langle \psi_x | Q_i | \psi_x \rangle \langle \psi_x | Q_j | \psi_x \rangle.$$

$\Rightarrow$ <u>distinguishing feature of phases!</u>

What about ground state energy? Degeneracy of $|\psi_\pm\rangle$ lifted – – what is splitting?

<u>Intuition:</u> $N$-th order perturbation theory: $\prod \sigma_i^x |\psi_\uparrow\rangle = |\psi_\downarrow\rangle$

$\rightarrow$ scales ~~as~~ $\cdot\, e^{-N}$. But: convergence unclear if $N \to \infty$!

$$|\langle\psi_+(s)|H_s|\psi_+(s)\rangle - \langle\psi_-(s)|H_s|\psi_-(s)\rangle| =$$

$$= |\langle\psi_+|H_s(s)|\psi_+\rangle - \langle\psi_-|H_s(s)|\psi_-\rangle|$$

$$= \left|2\,\mathrm{Re}\left[\langle\psi_+|H_s(s)|\psi_-\rangle\right]\right|$$

- $H_s(s)$ is exp. close to local.
- $\langle\psi_\uparrow|O|\psi_\downarrow\rangle \neq 0$ only of global $O$ (flip all spins)

$$= \underline{\underline{e^{-\ell/\xi}}}.$$

$\Rightarrow$ Exponentially small splitting.
(Note: Uses that $|\psi_\pm(s)\rangle$ are eigenstates.)

<u>Remark:</u> All this holds for any $H_s = \sum \sigma_i^z\sigma_j^z + V_s$, $[V_s, S] = 0$.

<u>Question:</u> What if $[V_s, S] \neq 0$?

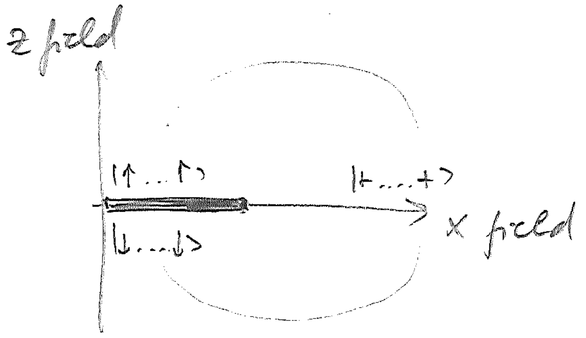
Consider e.g. $H_s = -\sum \sigma_i^z\sigma_j^z + s\sum \sigma_i^z$

$\rightarrow$ "infinite" splitting of ground states:

$$\langle\psi_\uparrow|H_s|\psi_\uparrow\rangle - \langle\psi_\downarrow|H_s|\psi_\downarrow\rangle = 2Ns \rightarrow \infty\,!$$

$\Rightarrow$ order not stable!

$\rightarrow$ $|\psi_{\uparrow\downarrow}\rangle$ eigenstates $\rightarrow$ no "true" long-range order, i.e. not different from $|+\ldots+\rangle$.

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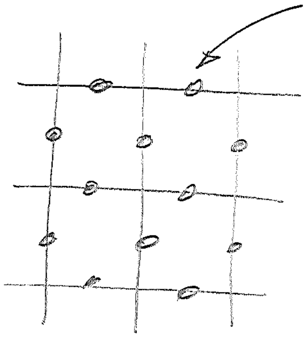
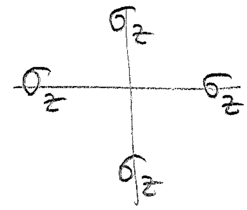
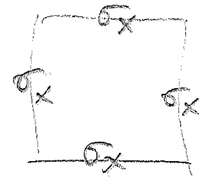
## Application 2: Topological phases

Toric Code model: 2D Lattice on torus




$$H_{TC} = -\sum_{v \in \text{vertices}} A_v - \sum_{p \in \text{plaquettes}} B_p$$

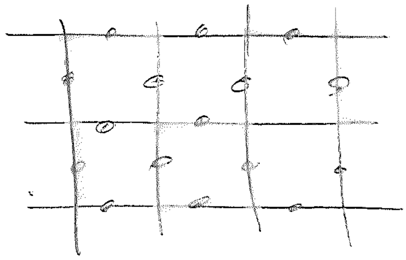
$A_v =$  $= \sigma_z^{\otimes 4}$ ; $B_p =$ 


We have $[A_v, A_{v'}] = [B_p, B_{p'}] = [A_v, B_p] = 0 \quad \forall v, v', p, p'$

$\Rightarrow$ ground state(s) of $H_{TC}$ joint eigenstates of all $A_v, B_p$!

Consider first $A_v$. Ground state = even # of $|1\rangle$'s around each vertex.

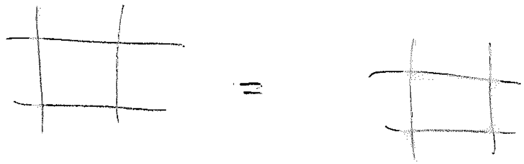
Picture: Mark $|1\rangle$ by "line":



ground space of $-\sum_v A_v$ consists of all loop patterns.

What is effect of $B_p$?

$B_p$ 

$B_p$  ... etc.

$B_p$ moves or creates loops.

+1-Eigenstates of $B_p$:

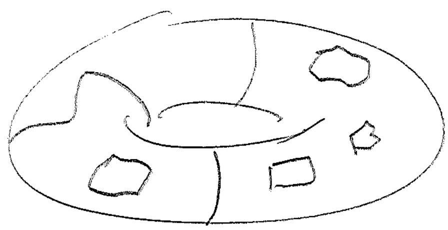
$B_p$  , etc.

$-B_p$ - Terms in $H$ favor "resonating" loop configurations: All configurations related by local $B_p$-moves must have same weight.

Guess: Ground state is uniform superposition of all loop patterns!

Is this the only ground state?

Consider superposition of all loop patterns with an even/odd # of loops winding around torus:



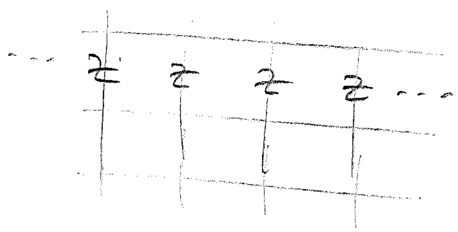
We can use local moves to create pairs of such global loops, but we cannot change their parity!

Holds for horizontal + vertical loops

$\longrightarrow$ 4 ground states (labelled by global parity).

These are the only ground states: All other loop configs connected by local moves. (This is not a proof, of course...)

How can we identify/distinguish different ground states?




$\xrightarrow[Z_L]{\text{loop operator}}$ counts the # of vertical loops mod 2.

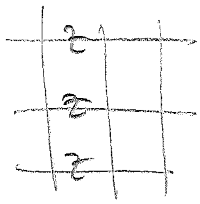
$(z = \sigma_z)$

(Local loops contribute 2 $\rightarrow$ counts global loops!)

$Z_h$ can be any loop: Can be moved by multiplication with $A_v$ (which has +1 eigenvalue) – does not change eigenvalue of $Z_h$.

$[Z_h, H_{TC}] = 0 \Rightarrow$ labels G.S.

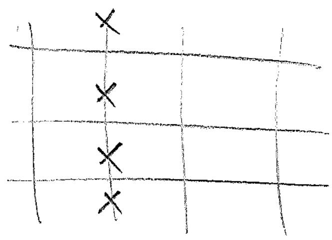
Similar:



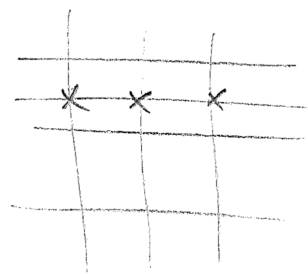

$\xrightarrow[Z_v]{\text{loop op.}}$ count horizontal loops mod 2.

$\rightarrow$ label 4 ground states $|\psi_{++}\rangle, |\psi_{+-}\rangle, |\psi_{-+}\rangle, |\psi_{--}\rangle$.

Can also define




$=: X_v$ ;




$=: X_h$.

$X_h, X_v$ create a non-trivial loop: they change betw. 4 G.S. above!

$\rightarrow$ full characterisation of ground space!

(Note: $X_h, X_v$ can also be moved anywhere.)

Overall:

$$[Z_a, H_{TC}] = [Z_v, H_{TC}] = [X_v, H_{TC}] = [X_a, H_{TC}] = 0.$$

$$\underbrace{\{Z_a, X_v\} = 0 \, ; \quad \{Z_v, X_a\} = 0}$$

generate Pauli algebra of 2 spin-$\frac{1}{2}$ (= qubits)

$\Rightarrow$ $Z_{a/v}$, $X_{a/v}$ fully characterize ground space (not only in a special basis). We can transform between any two ground states by evolution by $Z_{a/v}$, $X_{a/v}$.

Can we distinguish ground states locally?

Let $|\psi_\alpha\rangle, |\psi_\beta\rangle$ be any 2 G.S. with $|\psi_\alpha\rangle = L|\psi_\beta\rangle$ with $L$ a $\underbrace{}_{\text{unitary}}$ loop operator (generated by $X_{a/v}$, $Z_{a/v}$).

$L$ can be moved $\rightarrow$ for any local $O$, we can move $L$ such that $[L, O] = 0$.

$$\Rightarrow \langle\psi_\alpha|O|\psi_\alpha\rangle = \langle\psi_\beta|\cancel{L^\dagger} O \cancel{L}|\psi_\beta\rangle = \langle\psi_\beta|O|\psi_\beta\rangle.$$

$\Rightarrow$ ground states locally indistinguishable!

Can we transform ground states into each other by local operations?

Let $\langle\psi_\alpha|\psi_\beta\rangle = 0$:

$$\text{const} = \left(\langle\psi_\alpha| + e^{-i\phi}\langle\psi_\beta|\right) O \left(|\psi_\alpha\rangle + e^{+i\phi}|\psi_\beta\rangle\right)$$

$$= \underbrace{\langle\psi_\alpha|O|\psi_\alpha\rangle + \langle\psi_\beta|O|\psi_\beta\rangle}_{= \text{const.}} + \underbrace{2\mathrm{Re}\left[e^{i\phi}\langle\psi_\alpha|O|\psi_\beta\rangle\right]}_{\stackrel{!}{=} \text{const. (indep of } \phi)}$$

$$\Rightarrow \langle\psi_\alpha|O|\psi_\alpha\rangle = 0$$

$$\Rightarrow \langle\psi_\alpha|O|\psi_\beta\rangle = 0.$$

Properties of topo. order:

$$\langle\psi_\alpha|O|\psi_\alpha\rangle = \langle\psi_\beta|O|\psi_\beta\rangle \quad (\text{for \underline{arbitrary} } |\psi_\alpha\rangle, |\psi_\beta\rangle)$$

$$\langle\psi_\alpha|O|\psi_\beta\rangle = 0 \quad (\text{for } \langle\psi_\alpha|\psi_\beta\rangle = 0)$$

$\rightarrow$ related to "error correction conditions" in Quantum Information.

How is topo. order different from sym. breaking? Can e.g. Toric Code be connected to 4-fold degen. sym. broken state?

(Note: For topo. phases, degen. also depends on topology!)

$$\text{topo. order} \longleftrightarrow \text{sym. breaking}$$

$$4 \text{ G.S.} \quad \longleftrightarrow \text{path?} \quad 4 \text{ G.S.}$$

Quasi-adiabatic continuation:

Topological phase: $O$ Local:

$$\langle \psi_\alpha(s) | O | \psi_\alpha(s) \rangle = \langle \psi_\alpha | O(+s) | \psi_\alpha \rangle .$$

$O^\ell(+s)$ local s.th. $\| O^\ell(+s) - O(+s) \| \le O(e^{-\ell/\xi})$.

Note: We have ignored $\epsilon_{\alpha\beta}$, but it can be easily included $\rightarrow$ Homework

$$\Rightarrow \langle \psi_\alpha(s) | O | \psi_\alpha(s) \rangle = \langle \psi_\alpha | \underbrace{O^\ell(+s)}_{\rightarrow \text{const.}} + \underbrace{\left( O(+s) - O^\ell(+s) \right)}_{O(e^{-\ell/\xi})} | \psi_\alpha \rangle$$

$$= \text{const.} + O(e^{-\ell/\xi}).$$

and: $|\langle \psi_\alpha(s) | O | \psi_\beta(s) \rangle| = |\langle \psi_\alpha | O(+s) | \psi_\beta \rangle| \le O(e^{-\ell/\xi})$.

$\Rightarrow$ Conditions hold up to exp. small corrections!

## Compare w/ symm. breaking:



$\langle\psi_\alpha(s)|O|\psi_\alpha(s)\rangle \longrightarrow$ depends on G.S. (e.g. $\sigma_z$ on $|\uparrow\ldots\uparrow\rangle$ vs. $|\downarrow\ldots\downarrow\rangle$)

$\langle\psi_\alpha(s)|O|\psi_\beta(s)\rangle \neq 0$ dep. on state (e.g. $\sigma_z|\psi_+\rangle = |\psi_-\rangle$)

$\rightarrow$ behavior under local ops depends on ground state

($\rightarrow$ not useful to store quantum info.)

$\rightarrow$ distinct behavior $\Rightarrow$ gap must close, phase transition.

Energy splitting betw. G.S. in topo. phase?

$\rightarrow$ again exp. small for all $H_s$ local:

$H_s = \sum h_{z,s}$ is just a "special local observable".

Remaining question: Can we understand when gap remains open under arbitrary perturbations, i.e., when phases are stable?