# Exponential clustering of correlations

Consider spin system: ground state $|\psi_0\rangle$, $E_0|\psi_0\rangle = H|\psi_0\rangle$, or thermal state $\rho = \frac{1}{Z} e^{-\beta H}$.

Is the state "ordered" in some way – e.g. magnetized or anti-ferromagnetic, or are the spins independent?

Measure e.g. $\langle\psi_0|\sigma_z^i|\psi_0\rangle = m_i$ (or $tr[\rho\sigma_z^i]$)

(N.B.: We define shorthand $\langle O\rangle = \langle\psi_0|O|\psi_0\rangle$)

$m_i = \text{const.} \neq 0 \Rightarrow$ state magnetized

$m_i = (-1)^i \tilde{m} \neq 0 \Rightarrow$ antiferromagnet

$m_i = 0 \Rightarrow$ no order

"order parameter"

Esp. interesting: Will system be magnetized without external field?

I.e.: $H$ rotationally invariant ($SU(2)$ ~~or~~ $SO(3)$), e.g.

$$H = J \sum_{\langle ij\rangle} \vec{S}_i \cdot \vec{S}_j$$

Will the system order magnetically at some temperature (or $T=0$)?

"Problem": $\hat{H}$ $su(2)$-invariant (i.e., $[H, u^{\otimes N}]=0$ $\forall u$ unitary)

$\Rightarrow$ $\rho = e^{-\beta H}$ $SU(2)$-inv. for all $T$, and ground state

$|\psi_0\rangle$ as well (if it is unique).

$\Rightarrow$ $\langle\psi_0|\vec{S}_i|\psi_0\rangle = 0$ and $\mathrm{tr}[\rho \vec{S}_i] = 0$ due to symmetry!

(e.g. $\langle\psi_0|\sigma_i^z|\psi_0\rangle = \underbrace{\langle\psi_0|\sigma_x^{\otimes N}}_{=\langle\psi_0|} \overbrace{\sigma_i^z\, \sigma_x^{\otimes N}}^{=-\sigma_i^z} \underbrace{|\psi_0\rangle}_{=|\psi_0\rangle} = -\langle\psi_0|\sigma_i^z|\psi_0\rangle$ )

Is there a way to define / detect spontaneous ordering?

<u>One idea:</u> Perturb $H$ by small field $B$: $H'(B)\ H + B\sum\sigma_z$.

Consider $\rho'(B) = e^{-\beta H'(B)}$, or $|\psi_0'(B)\rangle$

$\Rightarrow$ will (potentially) have non-zero $m'(B) = \langle\psi_0'(B)|H'(B)|\psi_0'(B)\rangle$

Consider limit $\lim_{B\to 0^+} m'(B) = m_0$ $\nearrow$ $=0$: no order
$\searrow$ $\neq 0$: order ("spontaneous sym. breaking")

<u>Other idea:</u> Long-range order:

$\langle\psi|\sigma_z^i|\psi\rangle$, but maybe all spins are parallel w/ "random" direction?

$$\langle \psi_0 | \sigma_z^i \sigma_z^j | \psi_0 \rangle = \begin{cases} \to 0 \text{ as } d(i,j) \to \infty \\ \Rightarrow \text{spins indep.} \to \text{no order} \\ \to \text{const. as } d(i,j) \to \infty \\ \Rightarrow \text{"Long-range order"} \end{cases}$$

$\Rightarrow$ characteristic feature of ordered phases!

More generally:

"Connected correlation function"

$$C(A,B) = \langle \psi_0 | AB | \psi_0 \rangle - \langle \psi_0 | A | \psi_0 \rangle \langle \psi_0 | B | \psi_0 \rangle$$

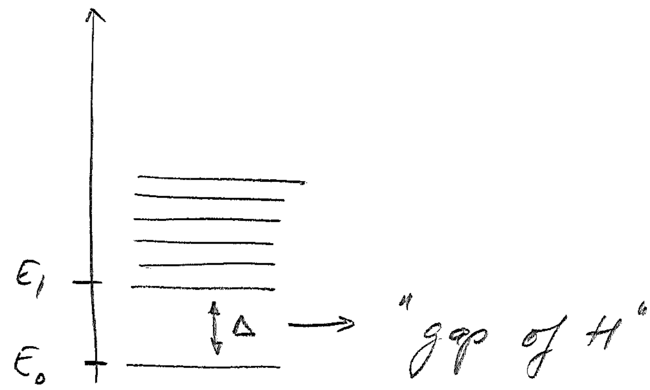
$\to$ only non-trivial long-range order

Can we find criteria for presence / absence of long-range order?

Will show: Long-range order related to properties in the eigenvalue spectrum of $H$ for local Hamiltonians!

Consider $H = \sum h_z$.

Eigenvalues:




We say that a (family of) Hamiltonians $H_N$ (e.g. the same $h_z$ on lattices of growing size) is gapped, if $\Delta(H_N)$ is lower bounded for $N \to \infty$, $\Delta(H_N) \geq \Delta > 0$.

Exponential clustering: For a gapped local Hamiltonian (w/ unique ground state), correlations decay exponentially with a length scale $\xi = \frac{2v}{\Delta}$, where $v$ is the LR-Robinson velocity.

Theorem (Exponential clustering; Hastings & Koma):

Let $H = \sum h_x$ have a unique ground state $|\psi_0\rangle$ and gap $\Delta$.
Let $v$ be such that an LR-bound holds,

$$\| [A_X, B_Y(t)] \| \leq \frac{v|t|}{\ell} g(\ell) \|A\| \|B\| \quad \text{for } v|t| < \ell,$$

where $\ell = d(X,Y)$ and $g(\ell) \sim e^{-\mu \ell}$!

Then

$$|\langle\psi_0|AB|\psi_0\rangle - \langle\psi_0|A|\psi_0\rangle\langle\psi_0|B|\psi_0\rangle| \leq \ldots$$

$$\ldots \leq \text{const.} \times \left(e^{-\ell\Delta/2v} + \min(|X|,|Y|)\,g(\ell)\right) \|A\|\,\|B\| \,.$$

(Note: similar version w/ degenerate ground states $|\psi_0^\alpha\rangle$ for

$$|\langle\psi_0^\alpha|AB|\psi_0^\alpha\rangle - \langle\psi_0^\alpha|A P_0 A|\psi_0^\alpha\rangle|\,, \quad P_0 = \sum|\psi_0^\alpha\rangle\langle\psi_0^\alpha|\Big)$$

Remark: This shows that in order to change the type of order, the gap of $H$ has to close.

Proof sketch: Use LR-bound!

(Note: For $\|[A,B(t)]\| \equiv 0$ – e.g. in relativistic theories – other proofs exist, see e.g. Fredenhagen, Commun. Math. Phys. 97, 461 (1985))

* Focus on unique G.S.

① Wlog, we can assume $\langle A\rangle = \langle B\rangle = 0$. Otherwise, replace $A \rightarrow A' = A - \langle A\rangle$, $B \rightarrow B' = B - \langle B\rangle$:

$$\langle AB\rangle - \langle A\rangle\langle B\rangle = \langle \underbrace{(A - \langle A\rangle)}_{A'}\underbrace{(B - \langle B\rangle)}_{B'}\rangle = \langle A'B'\rangle, \text{ and } \langle A'\rangle = \langle B'\rangle = 0.$$

(2) Rewrite $\langle\psi_0|AB|\psi_0\rangle$ in terms of $\langle\psi_0|[A,B^+]|\psi_0\rangle$:

* Define $B^+$: the "positive energy/frequency part" of $B$:

Energy eigenbasis of $H$: $(E_i, |\psi_i\rangle)$

$$(B^+)_{ij} \equiv \langle\psi_i|B^+|\psi_j\rangle = \langle\psi_i|B|\psi_j\rangle \cdot \Theta(E_i - E_j)$$

$$\left(\text{with } \Theta(\omega) = \begin{cases} 1, & \omega > 0 \\ 1/2, & \omega = 0 \\ 0, & \omega < 0 \end{cases}\right).$$

Then: $$\langle\psi_0|[A,B^+]|\psi_0\rangle = \underbrace{\langle\psi_0|A\underbrace{B^+|\psi_0\rangle}_{=B|\psi_0\rangle}}_{\text{use uniqueness of G.S.}} - \underbrace{\langle\psi_0|B^+}_{=0}A|\psi_0\rangle =$$

$$= \langle\psi_0|AB|\psi_0\rangle.$$

(Alternative approach: Take $\langle\psi_0|[A,B]|\psi_0\rangle \to$ decompose into Fourier components $e^{i\omega t}$ $\to \langle\psi_0|AB|\psi_0\rangle$ is the "pos. freq. part".)

3. Rewrite $B^+$ as integral over $B(t) = e^{iHt} B e^{-iHt}$

(later: $[A, B(t)] \to$ LR-bound!)

<u>Use:</u> a) $B(t)$ is <u>holomorphic</u>

b) $$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z} dz = \begin{cases} f(0), & 0 \text{ in } C \\ 0, & \text{otherwise} \end{cases}$$

for holomorphic $f$

(Cauchy formula / Residue theorem)

Consider for $\tau > 0$:

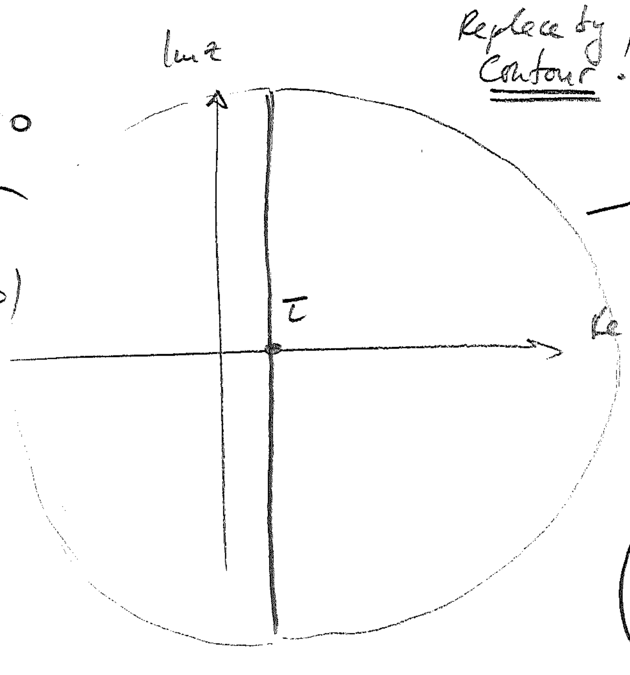
$$B^+(i\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, B(t) \frac{1}{it+\tau} = \sum_{ij} B_{ij} \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{i(E_i - E_j)t} \frac{1}{it+\tau}$$

$z = it + \tau$

$$\overset{\downarrow}{=} \sum_{ij} B_{ij} \frac{1}{2\pi i} \underbrace{\int_{-i\infty+\tau}^{+i\infty+\tau}}_{\substack{\text{Replace by} \\ \text{Contour!}}} dz\, \frac{e^{(E_i - E_j)(z-\tau)}}{z}$$

vanishes for $(E_i - E_j) > 0$

$\Rightarrow \frac{1}{2\pi i}\oint = f(0)$




vanishes for $(E_i - E_j) < 0$

$\Rightarrow \oint = 0$

(Neglect $E_i - E_j = 0$:
(i) not relevant, as $i$ or $j = 0$
(ii) still works!)

$$= \sum_{\substack{i,j \\ E_i > E_j}} B_{ij} \underbrace{e^{-(E_i - E_j)\tau}}_{\equiv f(0)} = e^{-\tau H} B^+ e^{\tau H}$$

$$\rightarrow \quad B^+ = \lim_{\tau \to 0^+} B^+(i\tau).$$

$$\Rightarrow \quad \boxed{B^+ = \frac{1}{2\pi} \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} dt \, B(t) \frac{1}{it + \tau}}$$

(Note: Can also be understood in terms of $\Theta$-Function → cf. later!)

④: Use ③ to bound commutator of ②:

$$|\langle \psi_0 | [A, B^+] | \psi_0 \rangle| \leq \frac{1}{2\pi} \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} dt \, \underbrace{|\langle \psi_0 | [A, B(t)] | \psi_0 \rangle|}_{\leq \underbrace{\| [A, B(t)] \|}_{\approx 0 \text{ for } |t| < \frac{\ell}{v}}} \frac{1}{|it + \varepsilon|}$$

* Integrand vanishes approximately for $|t| < \frac{\ell}{v}$ (due to LR-bnd.)
  → gives $g(\varepsilon)$-Term in Thm.

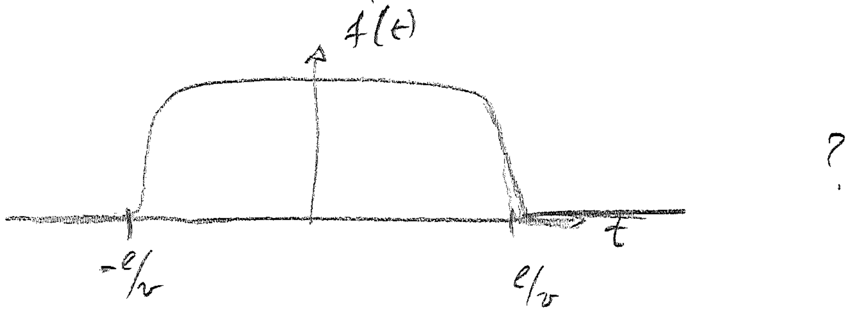
* Need <u>different bound</u> for $|t| \geq \frac{\ell}{v}$.

(5) <u>Idea</u>: Introduce filter function $\hat{f}(t)$:

$$\tilde{B}^{+} = \frac{1}{2\pi} \lim_{\varepsilon \to 0^{+}} \int dt \, B(t) \, \frac{1}{it+\varepsilon} \, \hat{f}(t)$$

Which properties should $\hat{f}(t)$ have?

(a) $\hat{f}(t)$ should be zero (or very small) outside $|t| \leq \frac{\ell}{v}$:




(b) $\tilde{B}^{+}$ should be close to $B^{+}$ — or, rather,

$\tilde{B}^{+}|\psi_0\rangle \approx B^{+}|\psi_0\rangle$ and $\langle\psi_0|\tilde{B}^{+} \approx \langle\psi_0|B^{+} = 0$.

$\Longrightarrow$ $\hat{f}(t)$ as flat as possible — i.e., its Fourier transform $f(\omega)$ should be localized in <u>frequency space</u>.

<u>I.e.</u>: $f$ should be localized in <u>time</u> and <u>frequency</u> space:

$\longrightarrow$ Lower bound imposed by <u>uncertainty relation</u>!

Moreover: Fast decay in time $\Longleftrightarrow$ slow decay in frequency and vice versa!

$\Rightarrow$ need to <u>balance</u> error of type ⓐ and ⓑ.

<u>Guess</u>: Take $\hat{f}(t)$ to be Gaussian (minimizes uncertainty!)

$$\underbrace{\hat{f}(t) = e^{-t^2/2\hat{\sigma}^2}}_{\text{Normalization } \hat{f}(0)=1} \longleftrightarrow f(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\omega^2/2\sigma^2} \; ; \; \sigma = \frac{1}{\hat{\sigma}}$$

$\Rightarrow$ For $\hat{\sigma} \to \infty$, $\tilde{B}^{+} = \tilde{B}$.

(<u>Note</u>: $\hat{g}(t) = \int_{\mathbb{R}} d\omega\, e^{-i\omega t} g(\omega)$; $g(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} dt\, e^{i\omega t} \hat{g}(t)$ )

Error of type ⓐ bounded by

$$2 \cdot \frac{1}{2\pi} \lim_{\tau \to 0^+} \int_{\ell/v}^{\infty} dt\, 2\|A\|\|B\| \frac{1}{|it+\tau|} e^{-t^2/2\hat{\sigma}^2} \le$$

$$\frac{2}{\pi} \frac{v}{\ell} \cdot \int_0^\infty dt'\, e^{-(\frac{\ell}{v}+t')^2/2\hat{\sigma}^2} \le$$

$$\frac{2v}{\pi\ell} e^{-(\ell/v)^2/2\hat{\sigma}^2} \underbrace{\int_0^\infty dt'\, e^{-\frac{\ell}{v}t'/\hat{\sigma}^2}}_{= -\frac{v\hat{\sigma}^2}{\ell} e^{-\frac{\ell}{v}t'/\hat{\sigma}^2}\Big|_0^\infty \le \frac{v\hat{\sigma}^2}{\ell}}$$

$$\le \frac{2v^2\hat{\sigma}^2}{\pi\ell^2} e^{-(\ell/v)^2/2\hat{\sigma}^2} \le \frac{2v^2\hat{\sigma}^2}{\underset{\text{const!}}{\pi}} e^{-(\ell/v)^2/2\hat{\sigma}^2}$$

Error of type ⓐ: const $\times\, e^{-(\epsilon/v)^2/2\hat{\sigma}^2}$.

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⑥ What is error of type ⓑ ( $\tilde{B}^+|\psi_0\rangle \overset{?}{\approx} B^+|\psi_0\rangle$, $\langle\psi_0|\tilde{B}^+ \overset{?}{\approx} 0$ )?

$$\left|(B^+ - \tilde{B}^+)|\psi_0\rangle\right|^2 = \sum_i \left|\langle\psi_i|(B^+ - \tilde{B}^+)|\psi_0\rangle\right|^2$$

eigenstates of $H$

$$|\langle\psi_i|B^+ - \tilde{B}^+|\psi_0\rangle| = \Big|\underbrace{(B^+)_{io}}_{=B_{io}} - \frac{1}{2\pi}\lim_{\tau\to 0^+}\int dt\, B_{io}\, e^{i(E_i - E_0)t}\frac{1}{it+\tau}\hat{f}(t)\Big|$$

$$= |B_{io}|\,\Big|1 - \frac{1}{2\pi}\lim_{\tau\to 0^+}\int dt\, e^{i(E_i-E_0)t}\frac{1}{it+\tau}\hat{f}(t)\Big|$$

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Excursion 1: What is $\lim\limits_{\tau\to 0^+} \ldots \dfrac{1}{it+\tau}$ ?

$\longrightarrow$ "Fourier transform of $\Theta$-Function"

Define $\Theta_\tau(\omega) := \Theta(\omega)\, e^{-\tau\omega}$ $\left(\Theta = \lim\limits_{\tau\to 0^+}\Theta_\tau\right)$

Then:

$$\hat{\Theta}_\tau(\omega) = \int \Theta_\tau(\omega) e^{-i\omega t} d\omega = \int_0^\infty e^{-\tau\omega} e^{-i\omega t} d\omega$$

$$= \left( \frac{-1}{it+\tau} e^{-\omega(it+\tau)} \right) \Big|_{\omega=0}^{\infty} = \underline{\underline{\frac{1}{it+\tau}}} .$$

Thus:

$$\frac{1}{2\pi} \int dt \; e^{i(E_i - E_0)t} \frac{1}{it+\tau} \hat{f}(t) = \underline{\underline{\mathcal{F}^{-1}\left[\hat{\Theta}_\tau \cdot \hat{f}\right](E_i - E_0)}}$$

Excursion 2: What is $\mathcal{F}^{-1}[\hat{g} \cdot \hat{h}]$ ?

(Sloppy!:)

$$\mathcal{F}^{-1}[\hat{g}\cdot\hat{h}](\omega) = \frac{1}{2\pi}\int e^{i\omega t} \hat{g}(t)\hat{h}(t) = \frac{1}{2\pi}\int dt\, d\nu\, d\mu \; e^{i\omega t} g(\nu) e^{-i\nu t} h(\mu) e^{-i\mu t}$$

$$= \int d\nu\, d\mu \; g(\nu) h(\mu) \underbrace{\left[ \frac{1}{2\pi} \int dt \; e^{i(\omega-\nu-\mu)t} \right]}_{= \delta(\omega-\nu-\mu) \;\Rightarrow\; \nu = \omega - \mu}$$

$$= \int d\mu \; g(\omega-\mu) h(\mu) =: (g * h)(\omega)$$

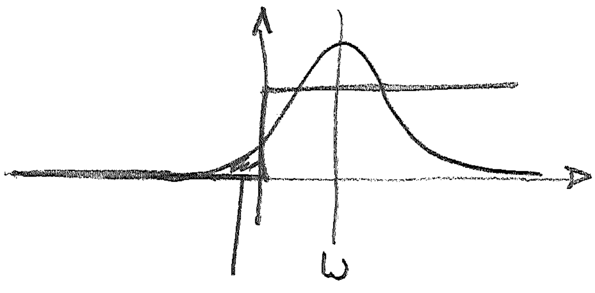
"Convolution" (German: "Faltung")

$$\Rightarrow \left| 1 - \underbrace{\frac{1}{2\pi} \lim_{\tau\to 0^+} \int dt\, e^{i(E_i - E_0)t} \underbrace{\frac{1}{it+\tau}}_{\hat{\Theta}_\tau(t)} \hat{f}(t)}_{= \lim_{\tau\to 0^+} (\Theta_\tau * f)(E_i - E_0)} \right|$$

$$= \left| 1 - (\Theta * f)(E_i - E_0) \right|$$

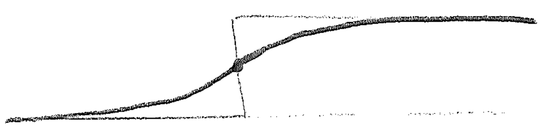
What is convolution (intuitively)?

$$\int d\nu\, \Theta(\omega-\nu) f(\nu) = \underbrace{\int d\nu\, \Theta(\omega+\nu) f(-\nu)}_{\Theta(\omega) \text{ "smeared out" by } f(-\nu).}$$




$(\Theta * f)(\omega)$

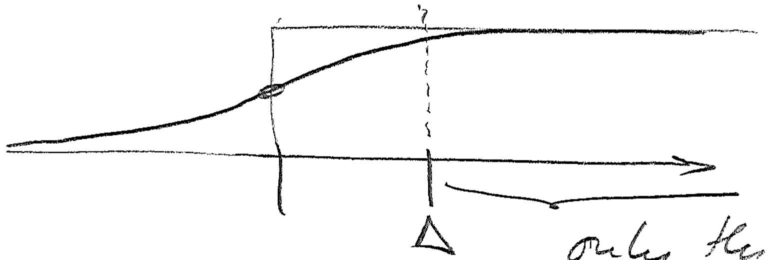
$\longrightarrow$



$(\Theta * f)(\omega)$ = tail of $f(\nu)$, $\nu \geq \omega$.

Generally: For good approx., $f(\omega)$ must be very narrow (we knew that...)

But: Gapped Hamiltonian $\Rightarrow$ $\underline{E_i - E_0 \geq \Delta}$ !



only this error matters!

$\Rightarrow$ $f(\omega)$ small outside $|\omega| \leq \Delta$ !

Concretely: weight in tail = error of type 6

$$= \int_\Delta^\infty d\omega f(\omega) = \int_\Delta^\infty \frac{d\omega}{\sqrt{2\pi\sigma^2}} e^{-\omega^2/2\sigma^2} \leq \ldots \leq \frac{\sigma^2}{\Delta} e^{-\Delta^2/2\sigma^2} .$$

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7 Summarize errors:

Type 2: $\sim e^{-(\epsilon/\omega)^2/2\hat{\sigma}^2}$ ; Type 6: $\sim e^{-\Delta^2/2\sigma^2}$ .

with $\sigma = \frac{1}{\hat{\sigma}}$

Best scaling of total error:

$$\frac{(\ell/v)^2}{\cancel{2}\hat{\sigma}^2} = \frac{-\Delta^2}{\cancel{2}\sigma^2}$$

$$\Rightarrow \quad \sigma^4 = \frac{\Delta^2 v^2}{\ell^2} \quad \Rightarrow \quad \sigma^2 = \frac{\Delta v}{\ell}$$

$\Rightarrow$ error scales as $e^{-\Delta^2/2\Delta v/\ell} = e^{-\Delta \ell/2v}$.

$\Rightarrow$ correlation length $\underline{\underline{\xi = \frac{2v}{\Delta}}}$.