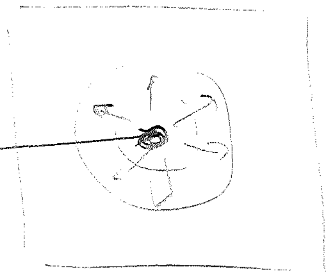
LRB – Robinson – Bounds

How does information propagate in spin systems?

perturbation at t=0



"speed of sound"?

Relativistic systems → speed of light.

Non-relativistic QM → a priori no such limit

LR: For spin systems on lattice, such limit exists!

Why interesting?

- Fundamental relevance: finite propagation speed w/out relativity
- Needed to define dynamics in the thermodyn. limit
- allows to transfer a range of "relativistic" results to non-rel. setting

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Definitions.

Lattice:

- Consider general lattice: Λ = set of lattice points
- Consider finite (but very big) lattice
- notion of distance (metric) $d(i,j): \Lambda \times \Lambda \longrightarrow \mathbb{R}_{\geq 0}$.

  (and Δ ineq.: $d(i,j) \leq d(i,k) + d(k,j)$)

E.g.: Square Lattice: $\Lambda = \{ i(i_1, i_2) : 0 \le i_1, i_2 \le N-1 \}$

$d(i,j)$: e.g. Euclidean distance, or "Manhattan metric",

possibly w/ periodic bnd.)

Define: For $X, Y \subset \Lambda$: $d(X,Y) = \min_{\substack{i \in X \\ j \in Y}} d(i,j)$

$$\mathrm{diam}(X) = \max_{i,j \in X} d(i,j)$$

$|X|$ = # of elements in X

Hilbert space: To each lattice site i associate $\mathcal{H}_i = \mathbb{C}^d$.

$$\mathcal{H}_X = \bigotimes_{i \in X} \mathcal{H}_i \;; \quad \mathcal{H}_\Lambda = \text{total H.S. of system}$$

Operators: $O : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$.

If $O = A_X \otimes \mathbb{1}_{\Lambda \setminus X}$, we say O is supported on X, $X = \mathrm{supp}\, O$.

Operator norm: $\|O\| = \max_{|\psi\rangle} \dfrac{\| O|\psi\rangle \|}{\| |\psi\rangle \|}$.

$$\|A \otimes B\| = \|A\| \, \|B\| \;; \quad \|\mathbb{1}\| = 1 \;; \quad \|A \cdot B\| \le \|A\| \, \|B\|$$

Hamiltonians:

$$H = \sum_{Z \subset \Lambda} h_Z \ , \text{ with } \operatorname{supp} h_Z = Z .$$

$$H_X = \sum_{Z \cap X \neq \emptyset} h_Z \ . \quad H \equiv H_\Lambda .$$

We will assume that H is <u>local</u> or <u>rapidly decaying</u>, i.e.,

$\|h_Z\|$ is zero/small if $\operatorname{diam}(Z)$ gets large.

<u>Theorem (Lieb-Robinson bound)</u>

Let H be such that there exist constants $\mu, s > 0$ s.th. for all $i \in \Lambda$

$$\sum_{X \ni i} \|h_X\| \, |X| \, e^{\mu \cdot \operatorname{diam}(X)} \leq s . \qquad (1)$$

Then, for any A_X, B_Y w. supp $A_X = X$, supp $A_Y = Y$, $d(X,Y) > 0$, it holds that

$$\| [A_X(t), B_Y] \| \leq 2 \|A_X\| \|B_Y\| \, |X| \, e^{-\mu d(X,Y)} \left(e^{2s|t|} - 1 \right) . \qquad (2)$$

Here, $O(t) = e^{iHt} O e^{-iHt}$ is the time-evolved op. in the Heisenberg picture.

Comments:

① What does (1) have to do with the locality of H?

- H local, e.g. 2-body Ham. w/ $\|h_X\| = J$ for all 2-body terms, and 2D square lattice:

$$\underbrace{\sum_{X \ni i}}_{4\times} \underbrace{\|h_X\|}_{=J} \underbrace{|X|}_{=2} e^{\mu \cdot \underbrace{\mathrm{diam}(X)}_{=1}} = \underline{\underline{8 J e^{\mu} =: s}}$$

→ Different pairs μ & s possible.

- $\|h_X\| \sim e^{-\gamma\, \mathrm{diam}(X)} \Rightarrow$ choose $\mu < \gamma$ & s accordingly

(2) What does (2) have to do w/ speed limit?

$$e^{-\mu d(X,Y)} \left(e^{2s|t|} - 1 \right) \le e^{-\mu \left(\underbrace{d(X,Y)}_{\text{distance}} - \underbrace{\frac{2s}{\mu}}_{\text{velocity}} |t| \right)}$$

prefactor.

By choosing μ we can optimize velocity. Above:

$$\frac{16 J e^{\mu}}{\mu} \text{ minimal for } \mu = 1 \Rightarrow \underline{\underline{v_{LR} = 16 \cdot e \cdot J}}$$

(3) What does the commutator "mean"?

$[A(t), B] = 0 \iff A(t)$ is only supported outside supp B.

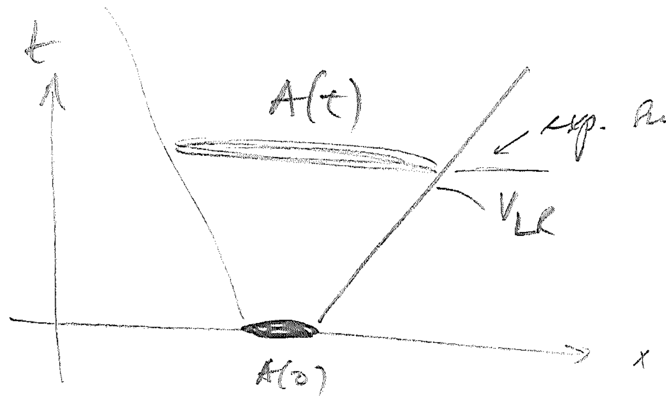
Therefore:

Let $K_\ell(X) = \{ i \mid d(i;X) \le \ell \}$, and B w/ supp in $\Lambda \setminus K_\ell(X)$.

$[A(t), B] = 0 \Longleftrightarrow A(t)$ supported in $K_\ell(X)$ $\Rightarrow$ effect of A cannot be seen outside that ℓ.

This statement "epsilonizes": i.e.: $d(X,Y) - \frac{2s}{\mu}|t| < 1$

$\Rightarrow$ $A(t)$ localized in region $\frac{2s}{\mu}|t|$.



Note: Very important that RHS of (2) indep. of $|Y|$!

Proof: 3 steps

Step 1: Express $\|[A(t), B]\|$ in terms of $\|[h_Z(s), B]\|$.

Step 2: Eliminate $A(t)$ by defining "worst case" commutator

$$C_B(X,t) = \sup_{A \in \mathcal{A}_X} \frac{\|[A(t), B]\|}{\|A\|}$$

algebra of all ops. w supp. on X.

Recursive expression for $G_B(X,t)$ $\Rightarrow$ series for $G_B(X,t)$ in terms of $\|h_Z\|$.

Step 3: Bound scattering decay of $\|h_Z\|$ — not used in Steps 1,2!

Step 1: $(t>0)$:

$$\|[A(t),B]\| - \|[A(0),B]\| = \sum_{n=0}^{N-1} \varepsilon \frac{\|[A(t_{n+1}),B]\| - \|[A(t_n),B]\|}{\varepsilon} = (*)$$

with $\varepsilon = \frac{t}{N}$, $t_n = \frac{t}{N} n = \varepsilon n$.

Now use that (i) $\|u^+ O u\| = \|O\|$, and (ii) $A(\varepsilon) = A + i\varepsilon[H_\Lambda, A] + O(\varepsilon^2)$:

$$\|[A(t_{n+1}),B]\| - \|[A(t_n),B]\| \overset{(i)}{=} \|[A(\varepsilon), B(-t_n)]\| - \|[A, B(-t_n)]\|$$

(w/ $u = \exp(iHt_n)$)

$$= \|[A + i\varepsilon \underbrace{[H_\Lambda, A]}_{\text{others commute!}} + O(\varepsilon^2), B(-t_n)]\| - \|[A, B(-t_n)]\|$$

$$\overset{\Delta\text{-ineq.}}{=} \|[A + i\varepsilon[H_X, A], B(-t_n)]\| - \|[A, B(-t_n)]\| + O(\varepsilon^2) = (**)$$

We can now use $A + i\varepsilon[H_X, A] = e^{i\varepsilon H_X} A e^{-i\varepsilon H_X} + O(\varepsilon^2)$:

$$\|[A + i\varepsilon[H_X, A], B(-t_n)]\| \leq \|[e^{i\varepsilon H_X} A e^{-i\varepsilon H_X}, B(-t_n)]\| + O(\varepsilon^2)$$

$$= \|[A, e^{-i\varepsilon H_X} B(-t_n) e^{i\varepsilon H_X}]\| + O(\varepsilon^2)$$

$$\le \| [A, B(-t_k) - i\varepsilon [H_X, B(-t_k)]] \| + O(\varepsilon^2)$$

$$\overset{\Delta\text{-ineq.}}{\le} \| [A, B(-t_k)] \| + \varepsilon \| [A, [H_X, B(-t_k)]] \| + O(\varepsilon^2).$$

In RHS of (**):

$$\| [A(t_{k+1}), B] \| - \| [A(t_k), B] \| \le \varepsilon \| [A, [H_X, B(-t_k)]] \| + O(\varepsilon^2).$$

$$\le 2\varepsilon \|A\| \cdot \| [H_X, B(-t_k)] \| + O(\varepsilon^2)$$

In (*):

$$\| [A(t), B] \| - \| [A(0), B] \| \le 2 \|A\| \sum_{k=0}^{N-1} \varepsilon \cdot \| [H_X(t_k), B] \| + O(\varepsilon)$$

$$H_X = \sum_{Z \cap X \neq \phi} h_Z$$

$$\le 2 \|A\| \sum_{Z \cap X \neq \phi} \sum_{k=0}^{N-1} \varepsilon \| [h_Z(t_k), B] \| + O(\varepsilon).$$

Limit $\varepsilon \to 0$ (as finite system):

$$\| [A(t), B] \| - \| [A(0), B] \| \le 2 \|A\| \sum_{Z \cap X \neq \phi} \int_0^t ds \, \| [h_Z(s), B] \|. \quad (***)$$

$\to$ We have re-expressed $\| [A(t), B] \|$ w/out any A (except for $\|A\|$).

Step 2: Define "worst case commutator" (normalized!)

$$C_B(X,t) := \sup_{A \in \mathcal{A}_X} \frac{\|[A(t), B]\|}{\|A\|}$$

$$(***) \Longrightarrow \quad C_B(X,t) \leq C_B(X,0) + 2 \sum_{Z \cap X \neq \phi} \|h_Z\| \int_0^t ds\, C_B(Z,s). \qquad (4*)$$

Will now solve (4*) iteratively.

Initial conditions: $\quad C_B(Z,0) = 0 \quad \text{if } Z \cap Y = \phi$

$$C_B(Z,0) \leq 2\|B\| \quad \text{if } Z \cap Y \neq \phi.$$

In particular: $X \cap Y = \phi \Rightarrow C(X,0) = 0.$

Iterative solution of (4*):

$$C_B(X,t) \leq \overbrace{C_B(X,0)}^{=0} + 2 \sum_{Z_1 \cap X \neq \phi} \|h_{Z_1}\| \int_0^t ds_1\, C_B(Z_1, s_1)$$

$$\leq 2 \cdot \sum_{Z_1 \cap X \neq \phi} \|h_{Z_1}\| \cdot \int_0^t ds_1 \cdot C_B(Z_1, 0)$$

$$+ 2^2 \cdot \sum_{Z_1 \cap X \neq \phi} \|h_{Z_1}\| \sum_{Z_2 \cap Z_1 \neq \phi} \|h_{Z_2}\| \int_0^t ds_1 \int_0^{s_1} ds_2\, C_B(Z_2, s_2)$$

$$\leq \ldots \text{ etc.} \qquad \leq$$

<u>NB</u>: Use that $C(Z,0) = \begin{cases} 2\|B\|, & Z\cap Y \neq \emptyset \\ 0, & \text{else} \end{cases}$

$$\to \sum_{Z_1}\int_0^t ds\, C_B(Z_1,0) = \sum_{Z_1\cap Y\neq\emptyset} t\cdot 2\|B\|$$

$$\sum_{Z_2}\int_0^t ds_1\int_0^{s_1} ds_2\, C_B(Z_2,0) = \sum_{Z_2\cap Y\neq\emptyset} \cdot\frac{t^2}{2!}\cdot 2\|B\|, \text{ etc.}$$

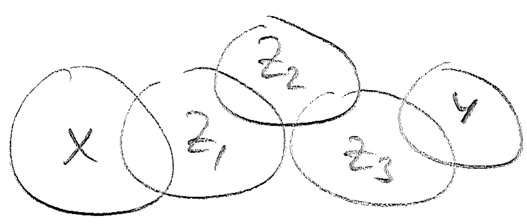
$$\ldots \leq 2\|B\| \frac{2t}{1!} \sum_{\substack{Z_1\cap X\neq\emptyset \\ Z_1\cap Y\neq\emptyset}} \|h_{Z_1}\| +$$

$$+ 2\|B\| \frac{(2t)^2}{2!} \sum_{Z_1\cap X\neq\emptyset} \|h_{Z_1}\| \sum_{\substack{Z_2\cap Z_1\neq\emptyset \\ Z_2\cap Y\neq\emptyset}} \|h_{Z_2}\|$$

$$+ 2\|B\| \frac{(2t)^3}{3!} \sum_{Z_1\cap X\neq\emptyset} \|h_{Z_1}\| \sum_{Z_2\cap Z_1\neq\emptyset} \|h_{Z_2}\| \sum_{\substack{Z_3\cap Z_2\neq\emptyset \\ Z_3\cap Y\neq\emptyset}} \|h_{Z_3}\|$$

$+ \ldots$

(Note: The h_{Z_i} connect X and Y:



)

Step 3: Bound the series using the decay of the $\|h_Z\|$.

(NB: For strictly local $\|h_Z\|$, this can be done using results from percolation theory, as $\underbrace{\sum\sum\sum\ldots}_{\ell} = \#\text{paths}\cdot\|h\|^{\ell}$)

Term by term:

1st term:

$$\sum_{\substack{Z_1\cap X\neq\emptyset\\ Z_1\cap Y\neq\emptyset}} \|h_{Z_1}\| \le \sum_{i\in X}\sum_{\substack{Z_1\ni i\\ Z_1\cap Y\neq\emptyset}} \|h_{Z_1}\| \overbrace{e^{\mu \underbrace{d(i;Y)}_{\le \operatorname{diam}(Z_1)}}\; e^{-\mu d(i;Y)}}^{=1}$$

$$\le \sum_{i\in X}\underbrace{\left(\sum_{\substack{Z_1\ni i\\ Z_1\cap Y\neq\emptyset}}\|h_{Z_1}\|\, e^{\mu \operatorname{diam}(Z_1)}\cdot\overbrace{|Z_1|}^{\ge 1}\right)}_{\text{decay: } \le s} e^{-\mu d(i;Y)}$$

$$\le s\cdot\sum_{i\in X} e^{-\mu d(i;Y)}$$

$\Rightarrow$ 1st term $= 2\|B\|\,\dfrac{(2ts)^1}{1!}\displaystyle\sum_{i\in X} e^{-\mu d(i;Y)}$

2nd term: Use $d(i;Y) \le d(i,j) + d(j,Y)$ (Δ-ineq.)

$$\Rightarrow\quad e^{-\mu d(i;Y)}\, e^{\mu d(i,j)}\, e^{\mu d(j,Y)} \ge 1.$$

Then:

$$\sum_{Z_1 \cap X \neq \phi} \|h_{Z_1}\| \sum_{\substack{Z_2 \cap Z_1 \neq \phi \\ Z_2 \cap Y \neq \phi}} \|h_{Z_2}\| \leq$$

$$\leq \sum_{i \in X} \sum_{Z_1 \ni i} \|h_{Z_1}\| \sum_{j \in Z_1} \sum_{\substack{Z_2 \ni j \\ Z_2 \cap Y \neq \phi}} \|h_{Z_2}\| \leq$$

$$\leq \sum_{i \in X} e^{-\mu d(i,Y)} \sum_{Z_1 \ni i} \|h_{Z_1}\| \sum_{j \in Z_1} e^{\mu d(i,j)} \underbrace{\sum_{\substack{Z_2 \ni j \\ Z_2 \cap Y \neq \phi}} \|h_{Z_2}\| \underbrace{e^{\mu d(j,Y)}}_{\leq e^{\mu \, \mathrm{diam}(Z_2)}}}_{\leq s}$$

$$\leq s \sum_{i \in X} e^{-\mu d(i,Y)} \underbrace{\sum_{Z_1 \ni i} \|h_{Z_1}\| \underbrace{\sum_{j \in Z_1} \underbrace{e^{\mu d(i,j)}}_{\leq e^{\mu \, \mathrm{diam}(Z_1)}}}_{\leq |Z_1| e^{\mu \, \mathrm{diam}(Z_1)}}}_{\leq s}$$

$$\leq s^2 \sum_{i \in X} e^{-\mu d(i,Y)}$$

$$\Rightarrow \underline{\text{2nd term}} = 2\|B\| \frac{(2ts)^2}{2!} \sum_{i \in X} e^{-\mu d(i,Y)}$$

etc. ...

Together:

$$C_B(X,t) \leq 2\|B\| \sum_{k\geq 1} \frac{(2ts)^k}{k!} \sum_{i\in X} e^{-\mu d(i,Y)}$$

$$= 2\|B\| \left(e^{2ts}-1\right) \sum_{i\in X} e^{-\mu d(i,Y)}$$

$$\boxed{d(i,Y) \geq \operatorname{dist}(X,Y)}$$

$$\leq 2\|B\| \left(e^{2ts}-1\right) e^{-\mu \operatorname{dist}(X,Y)} \cdot |X|$$