Problem 1

Let A^i and B^j MPS matrices such that their transfer operators are the same,

$$\sum A^i \otimes \bar{A}^i = \sum B^j \otimes \bar{B}^j \ .$$

Show that then there exists a partial isometry V (i.e., $VV^\dagger$ and $V^\dagger V$ are projectors) such that $B^j = \sum V_{ji} A^i$.

[*Note:* Show first that this is equivalent to having two matrices M and N such that $MM^\dagger = NN^\dagger$, and then use that any matrix has a *polar decomposition* (see e.g. on Wikipedia) $M = PU$ with U an isometry and P positive semi-definite matrix (i.e., a square hermitian matrix with non-negative eigenvalues).]

Problem 2

The AKLT state is defined by the matrices

$$A^{+1} = -\sqrt{2}\sigma^+ \ ; \qquad A^0 = \sigma_z \ ; \qquad A^{-1} = \sqrt{2}\sigma^- \ .$$

a) Show that there exists a basis transformation on the physical system (cf. Problem 1) such that the state is instead defined by the three Pauli matrices. Verify that the transfer operator obtained from both representations is the same.

b) Show that A translates a spin-1 representation of SU(2) to two spin-$\frac{1}{2}$ representations of SU(2), i.e.,

$$\sum S_{ij} A^j = A^i \, s - s \, A^i \ ,$$

where $S \in S_x, S_y, S_z$ is a generator of the spin-1 representation of SU(2) (i.e., a spin operator), and s is the corresponding spin-$\frac{1}{2}$ generator. Show that this implies that

$$\sum U_{ij} A^j = u^\dagger A u \ ,$$

where $U \equiv U_g$ and $u \equiv u_g$ are representations of $g \in$ SU(2) (i.e., spin rotations). (*Note:* Use that U can be expressed in terms of the generators, $U = \exp[\vec{\phi} \cdot \vec{S}]$.)

Show that this implies that the AKLT state (as a MPS with PBC) is invariant under the action of SU(2), i.e.,

$$|\Psi_{\mathrm{AKLT}}\rangle = U^{\otimes N} |\Psi_{\mathrm{AKLT}}\rangle$$

for any spin-1 rotation U.

Problem 3

In the lecture, we have discussed how the scaling of the correlations depends on the eigenvalues of the transfer operator. Find the corresponding formula for the expectation value of a single operator, and use this to determine the connected correlation functions

$$\langle AB \rangle - \langle A \rangle \langle B \rangle \ ,$$

and verify that the connected correlation functions go to zero exponentially (in the case of a single largest eigenvalue) or that they converge to a constant up to possible oscillations (in the case of a degenerate largest eigenvalue) constant.

As an example, consider the GHZ state with $A^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $A^1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and determine the behavior of the correlations at large distance. Compare this to the behavior observed for the "antiferromagnetic" state $A^0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $A^1 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.