Lecture *Entanglement and Locality in Quantum Many-body Systems* — Exercise Sheet #4

Problem 1

Show that the conditions derived for the ground states $|\psi_\alpha(s)\rangle$ of a Hamiltonian H_s which is connected to the toric code via a gapped path (i.e., via quasi-adiabatic continuation), namely

$$\langle\psi_\alpha(s)|O|\psi_\alpha(s)\rangle = \text{const.} + O(e^{l/\xi})$$
$$\langle\psi_\alpha(s)|O|\psi_\beta(s)\rangle = O(e^{-l/\xi}) \qquad (\text{ for } \langle\psi_\alpha|\psi_\beta\rangle = 0)$$

still hold when taking into account the rotation $G(s)$ of the ground subspace,

$$|\psi_\alpha(s)\rangle = \sum_\beta G_{\alpha\beta}(s)V(s)|\psi_\beta(0)\rangle \ .$$

Problem 2

a) Consider the horizontal loop operator Z_h in the toric code. Derive the operator $R_z(\phi) = \exp[-iZ_h\phi/2]$ which corresponds to a rotation of the encoded qubit about the z axis. Also compute the corresponding x rotation $R_x(\theta) = \exp[-iX_v\phi/2]$. Convince yourself that each of them is (i) unitary and (ii) can be moved arbitrarily by multiplying it with the Hamiltonian terms A_v and B_p.

b) Now consider the operation $U = R_z(\phi)R_x(\theta)R_z(\psi)$. (This can be used to realize an arbitary unitary on the qubit subspace.) Convince yourself that U is unitary, it is string-like, and it can still be moved arbitrarily by multiplying it with the A_v and B_p.

c) Show that for the Toric Code, it holds for any local operator O that

$$P_0OP_0 = c_OP_0 \ ,$$

where P_0 is the projector onto the ground space, and c_O is a constant which depends on O. Express c_O as a function of O and P_0.

Problem 3

In Lectures 7 and 8, we have used that a quasi-adiabatically evolved operator $O(s)$ can be approximated by $O^\ell(s)$ (supported on a region of size ℓ) up to accuracy $\sim e^{-\ell/\xi}$. However, this was not exactly what we derived.

a) Reconsider the derivation of the quasi-adiabatic evolution and its decay in Lecture 6. What is the decay we obtain from there?

b) In order to obtain a strictly exponential decay, we could have used a filter function F which is constructed from a Gaussian rather than from a smooth function g with compact support in Fourier space, following the proceduce of Problem 1 on Exercise Sheet 3. Show that this allows to obtain an error $\|O^\ell(s) - O(s)\|$ which is exponentially small in s. Is there also a price to pay for using Gaussians? How should we choose the trade-off in this case?
(*Note:* The idea of this part is not so much to carry out a rigorous calculation – though this is of course possible – but to identify the different sources of error, their scaling, and their relation.)