Lecture *Entanglement and Locality in Quantum Many-body Systems* — Exercise Sheet #2

Problem 1 (easy)

Verify that the integral formula given in the lecture for B^+,

$$B^+ = \frac{1}{2\pi} \lim_{\tau\to 0^+} \int_{-\infty}^{\infty} B(t) \frac{1}{it+\tau} \ ,$$

is also correct for degenerate eigenvalues, by showing that

$$\lim_{T\to\infty} \int_{-T}^{T} \frac{1}{it+\tau} = \pi \ ,$$

which is consistent with the convention $\theta(0) = \frac{1}{2}$ used in the definition of B^+. (Note that we need to consider the Cauchy principal value of the integral, since the integral diverges at $\pm\infty$.)

Problem 2 (varying difficulty: a medium, b tricky, c medium, d easy)

The goal of this problem is to generalize the Lieb-Robinson-bound to polynomially decaying interactions. The notation is the one used in the lecture.

a) Let $K(\ell) \geq 0$ (defined for $\ell \geq 0$) be a monotonically decreasing function which satisfies

$$\sum_{k\in\Lambda} K(d(i,k))K(d(j,k)) \leq \gamma K(d(i,j)) \ . \tag{1}$$

Let $H = \sum_{Z\subset\Lambda} h_Z$, such that

$$\sum_{Z\ni i,j} \|h_Z\| \leq K(d(i,j)) \ . \tag{2}$$

Show that then, a Lieb-Robinson-bound

$$\|[A(t),B]\| \leq \frac{2}{\gamma}\|A\|\,\|B\|\,|X|\,|Y|\,K(d(X,Y))\,(e^{2\gamma|t|}-1)$$

holds.

Hints: Note that you only need to modify Part 3 of the proof, i.e., the bound on the series on pg. 19 of the lecture notes. It is useful to expand each of the sums with auxiliary variables as in the lecture, but introducing yet an additional variable $j \in Y$, i.e.

$$\sum_{\substack{Z_1\cap X\neq\emptyset\\ Z_1\cap Y\neq\emptyset}} = \sum_{i\in X}\sum_{j\in Y}\sum_{Z_1\ni i,j} \ , \qquad \sum_{Z_1\cap X\neq\emptyset}\ \sum_{\substack{Z_2\cap Z_1\neq\emptyset\\ Z_2\cap Y\neq\emptyset}} = \sum_{i\in X}\sum_{j\in Y}\sum_{k\in\Lambda}\sum_{Z_1\ni i,k}\sum_{Z_2\ni k,j} \qquad \text{etc.}$$

b) Show that $K(\ell) = \frac{c}{(1+\ell)^\eta}$ satisfies Eq. (1), as long as $\eta > 0$ is large enough such that

$$\sum_{i\in\Lambda} K(d(i,j)) = \alpha < \infty \tag{3}$$

for all $j \in \Lambda$.

(*Hint:* Use – and prove – that $(1+d(i,j))^\eta \leq c_\eta[(1+d(i,k))^\eta + (1+d(k,j))^\eta]$ for some suitably chosen constant c_η. Otherwise, you can also check the proof in http://arxiv.org/abs/math-ph/0507008v3.)

c) Show that Eq. (3) holds if Λ is a regular D-dimensional hypercubic (i.e., square, cubic, etc.) lattice and $d(i,j) = \|i-j\|_2$ the 2-norm (i.e., euclidean) distance, as long as η is sufficiently large. What is the relation of η with the dimension D?

(*Note:* In finite dimensions all norms are equivalent, i.e., for any two norms $\|\cdot\|_x$ and $\|\cdot\|_y$ there exist c_1, c_2 such that $c_1\|\cdot\|_x \leq \|\cdot\|_y \leq c_2\|\cdot\|_x$. From this, one can show that (3) in fact holds for any distance defined through a norm. It also means that you can alternatively prove (3) using any other norm rather than the 2-norm.)

d) Show that for two-body interactions h_Z, $Z = \{i,j\}$, which decay as $\|h_Z\| \leq (d(i,j))^{-\mu}$, Eq. (2) holds with $K(\ell)$ as above. What is the relation between η and μ?