Last lecture:

Evolution of system cpl. to Markovian (= memoryless) environment:

Lindblad master equation

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \sum_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2} L_\alpha^\dagger L_\alpha \rho - \frac{1}{2}\rho L_\alpha^\dagger L_\alpha \right)$$

(L_α: Liouville operators)

Each L_α describes one "measurement by the environment"

$$\rho \longmapsto L_\alpha \rho L_\alpha^\dagger .$$

Example 1: 2-level atom cpl. to vacuum

$L_1 = \sqrt{\Gamma}\, \sigma^- \equiv$ emission of photon into environment

$$\dot{\rho} = -i \frac{\omega_{eg}}{2}[\sigma_z,\rho] - \frac{\Gamma}{2}\Big(\underbrace{\sigma^+\sigma^-\rho + \rho\sigma^+\sigma^-}_{\text{ensure normalization}} \underbrace{-2\sigma^-\rho\sigma^+}_{\text{loss at rate } \Gamma} \Big)$$

Write $\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix}$, $(\rho_{ge} = \rho_{eg}^*)$

$\Rightarrow$ Differential eqs. for matrix elements:

Independent equations for diagonal & off-diagonal terms:

$$\dot{\rho}_{ee} = -\Gamma \rho_{ee} \,; \qquad \dot{\rho}_{gg} = \Gamma \rho_{ee}$$

$$\dot{\rho}_{eg} = -i\omega_{eg} - \frac{\Gamma}{2}\rho_{eg}$$

$\Rightarrow$ ρ_{ee} decays exponentially to zero (at rate Γ)

$\rho_{gg} = 1 - \rho_{ee}$ $\quad (\leftrightarrow \mathrm{tr}\,\rho = 1)$

ρ_{eg} decays exp. to zero w/ oscillation

Interpretation on Bloch sphere:

Point in Bloch sphere decays to $|g\rangle$ at rate Γ & rotates about z axis w/ freq. ω_{eg}.

Interpretation of terms in master eq. (in terms of Kraus operators):

Imagine we monitor environment in time steps dt for photons:

- <u>Photon detected</u> $\Rightarrow M_1 = L_1\sqrt{dt} = \sqrt{\Gamma dt}\,|g\rangle\langle e|$ implemented

  $\Rightarrow$ state of system "jumps" from $|e\rangle$ to $|g\rangle$ ("quantum jump")

  Prob. for this event is $\mathrm{tr}[M_1 \rho M_1^\dagger] \equiv \Gamma dt \langle e|\rho|e\rangle$.

- <u>No photon detected</u>: Π_0 implemented, i.e., system evolves during time dt according to

$$\dot{\rho}(dt) = -\frac{i}{\hbar}[H,\rho] - \frac{1}{2}\sum\left(L_\alpha^+ L_\alpha \rho + \rho L_\alpha^+ L_\alpha\right)$$

$$= -i\frac{\omega_{eg}}{2}[\sigma_z,\rho] - \frac{\Gamma}{2}\left(\sigma^+\sigma^-\rho + \rho\sigma^+\sigma^-\right)$$

$$\Rightarrow \quad \dot{\rho}_{ee} = -\Gamma\rho_{ee}$$
$$\dot{\rho}_{gg} = 0$$
$$\dot{\rho}_{eg} = -i\omega\rho_{eg} - \frac{\Gamma}{2}\rho_{eg}$$

This is the <u>unnormalized conditional evolution</u> given no photon was observed
$\Rightarrow$ it is <u>not trace preserving</u>

$\Rightarrow$ ρ_{ee} decays exp. to zero, ρ_{gg} is constant

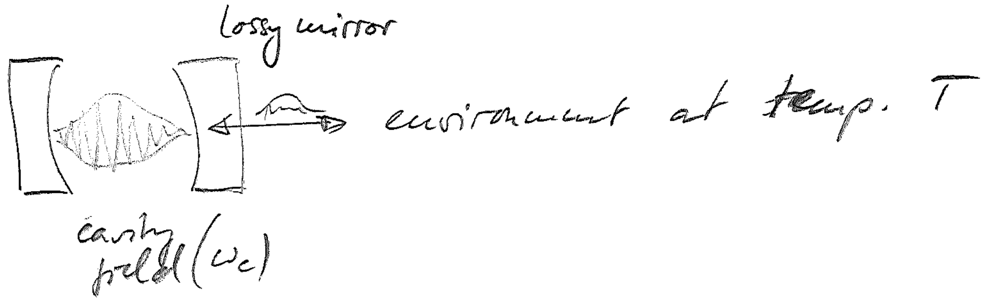
$\Rightarrow$ if system was initially in superposition/mixture of $|g\rangle$ and $|e\rangle$, this increases the relative prob. that system is in $|g\rangle$

$\Rightarrow$ Seeing <u>no photon</u> <u>increases our knowledge</u> about the state of the system & this leads to non-trivial evolution given by Π_0.

<u>Microscopic model</u>: $\Gamma = \dfrac{d^2\omega_{eg}^3}{3\pi\varepsilon_0\hbar c^3}$,

with d dipole moment of transition (cf. HW)

Example 2: Master eq. for cavity cpl. to thermal environment



Two possible Lindblad operators:

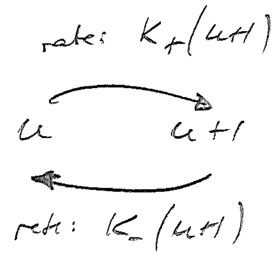
$L_- = \sqrt{\kappa_-}\, a$: loss of photon from cavity to environment

$L_+ = \sqrt{\kappa_+}\, a^\dagger$: absorption of photon from environment

Relation of κ_+ and κ_-:

$L_- \longrightarrow$ energy of env. increases by $\hbar\omega_c$

$L_+ \longrightarrow$ energy of env. decreases by $\hbar\omega_c$



Ratio of rates must be equal to ratio of probabilities of finding env. in states w/ energy E and $E+\hbar\omega_c$!

$$\frac{\kappa_-}{\kappa_+} = e^{\hbar\omega_c/k_BT} = \frac{1+n_{th}}{n_{th}}$$

Planck distr.: $n_{th} = \dfrac{1}{e^{\hbar\omega_c/k_BT}-1}$ (→ HW)

$$\Rightarrow \kappa_- = \kappa\,(n_{th}+1)$$

$$\kappa_+ = \kappa\, n_{th}$$

(80)

$$\frac{d\rho}{dt} = -i\omega_c\left[a^\dagger a, \rho\right] - \frac{\kappa(n_{th}+1)}{2}\left(a^\dagger a\rho + \rho a^\dagger a - \underbrace{2a\rho a^\dagger}_{\text{loss of photon}}\right)$$

$$- \frac{\kappa n_{th}}{2}\left(a a^\dagger \rho + \rho a a^\dagger - \underbrace{2a^\dagger \rho a}_{\text{absorption of photon}}\right)$$

Interaction picture ($a \mapsto a e^{-i\omega_c t}$, $a^\dagger \mapsto a^\dagger e^{i\omega_c t}$): remove Ham. part, leave remaining terms unchanged.

Express ρ in Fock basis, $\rho = \sum_{n,m\geq 0} \rho_{n,m} |n\rangle\langle m|$:

$$\dot\rho_{n,m} = -\frac{\kappa(n_{th}+1)}{2}\left((n+m)\rho_{n,m} - 2\sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1}\right)$$

$$- \frac{\kappa n_{th}}{2}\left((n+1+m+1)\rho_{n,m} - 2\sqrt{n\cdot m}\,\rho_{n-1,m-1}\right)$$

If we start in Fock state, $\rho_{n,n} = 1$, and $\rho_{k,l} = 0$ otherwise:

Only diagonal $\rho_{n,n}$ are ever non-zero $\Rightarrow$ Diff.-Eq. for occupation prob. $p(n) = \rho_{n,n} = \langle n|\rho|n\rangle$:

$$\dot p(n) = \underbrace{\kappa(n_{th}+1)(n+1)}_{\text{Rate } n+1 \to n} p(n+1) + \underbrace{\kappa n_{th} n}_{\text{Rate } n-1 \to n} p(n-1)$$

$$- \underbrace{\kappa\left((n_{th}+1)n + n_{th}(n+1)\right)}_{\text{Rate } n \to n\pm 1} p(n)$$

Fixed pt. solution:

Detailed balance, i.e. Rate$(n \to n-1) \cdot p(n) =$ Rate$(n-1 \to n) \cdot p(n-1)$

$$\Rightarrow \quad \frac{p(n)}{p(n-1)} = \frac{\kappa n_{th}\, n}{\kappa (n_{th}+1)\, n} = \frac{n_{th}}{1+n_{th}} = e^{-\hbar\omega_c/kT}$$

$\to$ Boltzmann distr. $\Rightarrow$ system equilibrates to thermal state.

(Note: This holds for any initial state, as off-diag. terms vanish.)

Relaxation rate:

$$\langle n \rangle = \sum n\, p(n)$$

$$\frac{d}{dt}\langle n \rangle = \sum n \frac{d}{dt} p(n) = \overset{(HW)}{\ldots} = \kappa \left(n_{th} - \langle n \rangle \right)$$

Relaxation to thermal photon # at rate $\kappa = \frac{\omega_c}{Q}$ (Q: "quality factor")

Relaxation characterized by 2 parameters only:

$\kappa \mathrel{\hat{=}}$ cavity quality factor / damping

$n_{th} \mathrel{\hat{=}}$ temperature of environment (rel. to $\hbar\omega_c$)

Similarly: cpl. of 2-level atom to thermal env.:

$$L_+ = \sqrt{\Gamma n_{th}}\, \sigma^+, \quad L_- = \sqrt{\Gamma (1+n_{th})}\, \sigma^- \quad \text{w/ same } \Gamma.$$

$n_{th}+1$ vs. n_{th}: spont. emission in vacuum (follows from fairly general thermodyn. considerations!)

Example 3: Atom + cavity cpl. to environment — the Purcell effect

Atom + cavity w/ losses to (thermal) environment:

$$\dot{\rho} = -\frac{i}{\hbar}\left[H_a + H_c + H_{ac}, \rho\right] + \sum_i \left[L_i \rho L_i^\dagger - \frac{1}{2}\left(L_i^\dagger L_i \rho + \rho L_i^\dagger L_i\right)\right]$$

$H_a = \frac{1}{2}\hbar\omega_{eg}\sigma_z$: atom; $H_c = \hbar(a^\dagger a + \frac{1}{2})\omega_c$: cavity;

$H_{ac} = -i\hbar\frac{\Omega_0}{2}(a\sigma^+ - a^\dagger\sigma^-)$: atom-cavity-cpl. (J-C model)

$L_1 = \sqrt{\kappa(n_{th}+1)}\, a$; $L_2 = \sqrt{\kappa n_{th}}\, a^\dagger$: cavity - env. cpl.

$L_3 = \sqrt{\Gamma(n_{th}+1)}\,\sigma^-$; $L_4 = \sqrt{\Gamma n_{th}}\, a^\dagger$: atom - env. cpl.

Focus on resonant case, $\omega_{eg} = \omega_c$.

Different regimes dep. on relative time scales $\frac{1}{\Gamma}, \frac{1}{\kappa}, \Omega_0$.

Simplest case: vacuum ($T=0$) env., and $1/\Gamma \gg 1/\kappa, \Omega_0$ (e.g. for Rydberg atoms)

Interaction picture w.r.t. $H_a + H_c$:

$$\dot{\rho} = -\frac{i}{\hbar}\left[H_{ac}, \rho\right] + L\rho L^\dagger - \frac{1}{2}\left(L^\dagger L\rho + \rho L^\dagger L\right); \quad L = L_1 = \sqrt{\kappa}\, a$$

Two regimes expected:

$\Omega_0 \gg \kappa$: weakly damped Rabi oscillations

$\Omega_0 \ll \kappa$: strongly damped: fast decay of atom via cavity

Solution of model:

Three levels occur: $\overset{\text{atom}}{\downarrow}\ \overset{\text{cavity}}{\downarrow}$

$$|e,0\rangle \equiv |1\rangle$$

$$|g,1\rangle \equiv |2\rangle$$

$$|g,0\rangle \equiv |3\rangle$$

$$\rho = \sum \rho_{ij} |i\rangle\langle j| .$$

Initial state $\rho(t=0) = |1\rangle\langle 1|$

Possible transitions:

$$\begin{tikzcd}
 & |1\rangle\langle 2| \arrow[dr, leftrightarrow, "H_{ac}"] & & \\
|1\rangle\langle 1| \arrow[ur, leftrightarrow, "H_{ac}"] \arrow[dr, leftrightarrow, "H_{ac}"'] & & |2\rangle\langle 2| \arrow[r, "L"] & |3\rangle\langle 3| \\
 & |2\rangle\langle 1| \arrow[ur, leftrightarrow] & &
\end{tikzcd}$$

$\Rightarrow$ only 3 indep. parameters: $\rho_{11}, \rho_{22}, \rho_{12} = \rho_{21}^{*}$ ($\rho_{33} = 1 - \rho_{11} - \rho_{22}$!)

$$\boxed{\begin{aligned} \dot\rho_{11} &= -\frac{\Omega_0}{2}\left(\rho_{12} + \rho_{21}\right) \\ \dot\rho_{22} &= \frac{\Omega_0}{2}\left(\rho_{12} + \rho_{21}\right) - \kappa\rho_{22} \end{aligned}}$$

and $\dot\rho_{12} = \frac{\Omega_0}{2}\left(\rho_{11} - \rho_{22}\right) - \frac{\kappa}{2}\rho_{12}$

$$\Rightarrow \boxed{\rho_{12} + \rho_{21} = 2\,\mathrm{Re}\,\rho_{12} = \Omega_0\left(\rho_{11} - \rho_{22}\right) - \frac{\kappa}{2}\left(\rho_{12} + \rho_{21}\right)}$$

and $\rho_{12} - \rho_{21} = 2i\,\mathrm{Im}\,\rho_{12} = -\kappa\,\underbrace{\left(\rho_{12} - \rho_{21}\right)}_{=0 \text{ at } t=0}$ $\Rightarrow$ stays zero!

First-order differential eq. in 3 variables:

$$\frac{d}{dt}\begin{pmatrix} \rho_{11} \\ \rho_{22} \\ \rho_{12}+\rho_{21} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & -\frac{\Omega_0}{2} \\ 0 & -\kappa & \frac{\Omega_0}{2} \\ \Omega_0 & -\Omega_0 & -\kappa/2 \end{pmatrix}}_{M} \begin{pmatrix} \rho_{11} \\ \rho_{22} \\ \rho_{12}+\rho_{21} \end{pmatrix}$$

Solution from eigenvalues & eigenvectors of M.

Eigenvalues: $\lambda_0 = -\frac{\kappa}{2}$; $\lambda_\pm = -\frac{\kappa}{2}\left(1 \pm \sqrt{1-\frac{4\Omega_0^2}{\kappa^2}}\right)$

Two regimes;

$\boxed{\kappa < 2\Omega_0}$ oscillatory regime:

Rabi oscillations w. freq. $\Omega_0\sqrt{1-\frac{\kappa^2}{4\Omega_0^2}}$ $\left(\approx \Omega_0 \text{ for } \Omega_0 \gg \kappa\right)$

and damping w cavity damping rate κ.

$\boxed{\kappa > 2\Omega_0}$ Overdamped regime

Over-damped oscillations. Decay rate (for $\kappa \gg 2\Omega_0$):

$$\Gamma_c = -\lambda_- = \frac{\kappa}{2}\left(1-\sqrt{1-\frac{4\Omega_0^2}{\kappa^2}}\right) \underset{\kappa \gg \Omega_0}{\approx} \frac{\Omega_0^2}{\kappa}$$

Compare to vacuum decay rate:

$$\Gamma_c = \frac{\Omega_0^2}{\kappa} = \frac{2d^2}{\varepsilon_0 \hbar}\,\frac{Q}{V} \qquad \left(\text{as } \Omega_0^2 = \frac{2d^2\omega_c}{\varepsilon_0 \hbar V}\,;\; \kappa = \frac{\omega_c}{Q}\right)$$

$$\Gamma_{vac} = \frac{d^2\omega_c^3}{3\pi\varepsilon_0 \hbar c^3}$$

$$\Rightarrow \Gamma_c = \eta\,\Gamma_{vac} \quad \text{with} \quad \underline{\underline{\eta = \frac{3}{4\pi^2}\,\frac{Q\lambda_c^3}{V}}}\,;\; \lambda_c = \frac{2\pi c}{\omega_c}$$

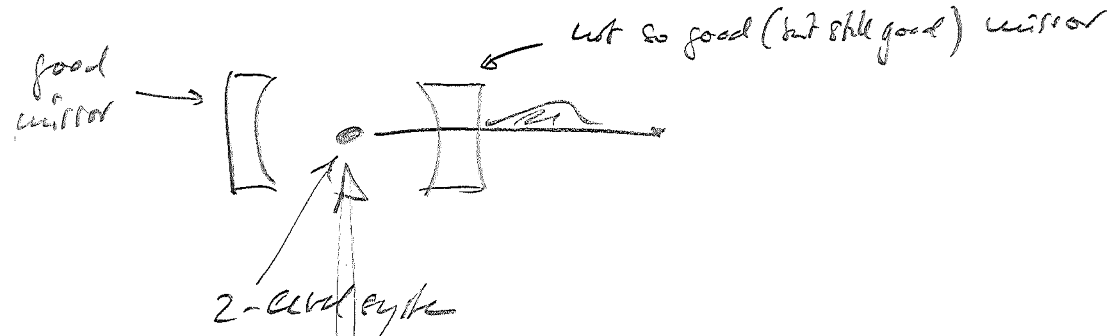
Spontaneous emission <u>enhanced</u> by factor η!

large enhancement $\longleftrightarrow$ large Q, small V (rel. to λ_c)

(But: limited by $\kappa > 2\Omega_0 \leftrightarrow \frac{\omega_c}{2\Omega_0} > Q$)

<u>"Purcell effect"</u>: Presence of cavity <u>enhances</u> spont. emission.

<u>Use:</u> Generation of single photons on demand:



excite atom $\Rightarrow$ atom will w/ very high prob. emit into <u>one</u> mode!