Last lecture: Mixed states

Pure state: vector $|\psi\rangle \in \mathcal{H}$ (Hilbert space)

observable $O \Rightarrow$ exp. value $\langle O \rangle = \langle \psi | O | \psi \rangle$

Mixed state: linear map (= matrix) $\rho : \mathcal{H} \to \mathcal{H}$.
(Can be written in basis $|n\rangle$: $\rho = \sum_{n,m} \rho_{n,m} |n\rangle\langle m|$)

observable $O \Rightarrow$ exp. value $\langle O \rangle = \mathrm{tr}[O\rho]$
(Note: trace is cyclic in finite dim.: $\mathrm{tr}[AB] = \mathrm{tr}[BA]$)

For pure state: $\rho = |\psi\rangle\langle\psi|$.

Mixed state $\rightarrow$ incomplete knowledge:

- statistical mixture:
  state $|\psi_i\rangle$ w/ prob. $p_i \Rightarrow \rho = \sum p_i |\psi_i\rangle\langle\psi_i|$

- part of the system ("environment") inaccessible:
  bipartite state $|\psi\rangle_{SE} = \sum w_{ij} |\alpha_i\rangle_S |\beta_j\rangle_E$
  $\Rightarrow$ reduced state of S is

$$\rho_S = \mathrm{tr}_E [|\psi\rangle_{SE}\langle\psi|_{SE}] = \sum w^*_{i'j'} w_{ij} \langle\beta_{j'}|\beta_j\rangle_E |\alpha_i\rangle_S \langle\alpha_{i'}|_S$$

- no unique interpretation of mixed state (different from classical statistical ensemble!)

- interpretations equivalent $\Leftrightarrow$ we can choose most convenient interpretation.

iii) Channels & measurements:

- Unitary evolution: $\rho \to U \rho U^{\dagger}$ (e.g.; $U = e^{-iHt/\hbar}$)

- Projective measurement:

  Projectors $E_\alpha = E_\alpha^{\dagger}$; $E_\alpha^2 = E_\alpha$; $\sum E_\alpha = \mathbb{1}$:

  $p(\alpha) = \mathrm{tr}[E_\alpha \rho]$ $(\hat{=} \langle\psi|E_\alpha|\psi\rangle)$ prob. for outcome α

  State after measurement is

  $$\rho(\alpha) = \frac{1}{p(\alpha)} \left(E_\alpha \rho E_\alpha^{\dagger} \right)$$

- General measurement: System interacts w/ meas. apparatus E, and we perform (proj.) measurement on E.

  - Interaction: $\rho_S \otimes \underset{\substack{\uparrow \\ \text{initial state of apparatus}}}{|0\rangle\langle 0|_E} \xrightarrow{U_{SE}} U_{SE} \left(\rho_S \otimes |0\rangle\langle 0|_E \right) U_{SE}^{\dagger}$

Assume E_α has rank 1: $E_\alpha = |\mu_\alpha\rangle\langle\mu_\alpha|$.

(Higher rank can be realized by discarding part of meas. outcome.)

Measurement of E:

$$\underline{p(\alpha)} = \mathrm{tr}_{SE}\left[|\mu_\alpha\rangle_E\langle\mu_\alpha|_E\, U_{SE}\,(\rho_S\otimes|0\rangle_E\langle 0|_E)\,U_{SE}^\dagger\right]$$

$$= \underline{\mathrm{tr}_S\left[M_\alpha\rho M_\alpha^\dagger\right]}; \quad \text{with } M_\alpha = \underbrace{\langle\mu_\alpha|_E\, U_{SE}\,|0\rangle_E}_{\text{matrix acting on } S.}$$

$$\rho(\alpha) = \frac{1}{p(\alpha)} M_\alpha\rho M_\alpha^\dagger$$

It holds that

$$\underline{\sum M_\alpha^\dagger M_\alpha} = \sum_\alpha \langle 0|_E U_{AB}^\dagger \underbrace{|\mu_\alpha\rangle_E\langle\mu_\alpha|}_{=\mathbb{1}} U_{AB}|0\rangle_E = \langle 0|_E \mathbb{1}_{SE}|0\rangle_E = \underline{\mathbb{1}_S}$$

$$\left(\Rightarrow \text{implies } \sum p(\alpha) = \mathrm{tr}\left(\sum_\alpha M_\alpha\rho M_\alpha^\dagger\right) = \mathrm{tr}\left(\rho\sum M_\alpha^\dagger M_\alpha\right) = \mathrm{tr}(\rho)\right)$$

$\{M_\alpha\}$ s.th. $\sum M_\alpha^\dagger M_\alpha = \mathbb{1}$ is called Positive Operator Valued Measure (POVM).

Conversely: Any POVM measurement $\{M_\alpha\}$ can be realized by adding an "ancilla" (= environment) E, acting on system + ancilla w/ some U, and measuring E.

* evolution:

* Hamiltonian evolution: $U = e^{-iHt/\hbar}$; $\rho = U\rho U^+$.

* most general evolution: couple system to environment E, let it interact via U_{SE}, and discard (= trace out) E.

$\Rightarrow$ identical to POVM measurement, but discarding the meas. outcome!

$$\Rightarrow \rho \rightarrow \underline{T(\rho)} = \sum p(\alpha) \frac{1}{p(\alpha)} \Pi_\alpha \rho \Pi_\alpha^+$$

$$= \underline{\underline{\sum \Pi_\alpha \rho \Pi_\alpha^+}}\ , \text{ with } \sum \Pi_\alpha^+ \Pi_\alpha = \mathbb{1}$$

"<u>Kraus representation</u>" of quantum channel T.

<u>Note</u>:
- Any such channel can be realized by unitary cpl. to some environment.
- Any physical map $\rho \mapsto T(\rho)$ (hermitian, trace preserving, and completely positive) is of this form!
- We can picture that the environment E "knows" which α has been realized (i.e., we could infer α by measuring E)

iii) Continuous evolution:

* Hamiltonian evolution: $\rho(t) = e^{-iHt/\hbar}\rho_0 e^{iHt/\hbar} \Longrightarrow \dot{\rho} = -\frac{i}{\hbar}[H,\rho]$.

* How can we differentially describe evolution of a system coupled to an environment?

Problem: System & environment become entangled ($\hat{=}$ exchange information) — No closed description of system exists!

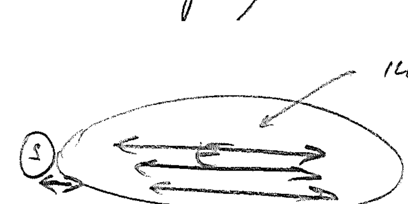
However; In many cases, environment is very large & "forgets" information very quickly (forget $\hat{=}$ does not affect evol. of syst.)

E.g.: Atom emits photon into free space $\rightarrow$ photon "is gone" and does not affect atom any more.

Markov approximation:

τ_{env}: time scale at which environment "looses information"

τ_{sys}: time scale of system/environment coupling.



Discretize evolution in time steps dt: $\tau_{sys} \gg dt \gg \tau_{env}$:

∘ Bath in steady state + system evolves only by small amount:

$$\rho(t+dt) \approx \mathcal{L}[\rho(t)] \text{ only! (indep. of E)}$$

$\mathcal{L}$ can be obtained by taking Kraus representation

$$\rho \longrightarrow \sum M_\alpha \rho M_\alpha^+ \; ; \; \sum M_\alpha^+ M_\alpha = \mathbb{1}$$

and considering differential evolution:

$$M_0 = \mathbb{1} + \left(\frac{-i}{\hbar} H + K\right) dt \qquad \left(\text{with } H = \frac{i\hbar (M_0 - M_0^+)}{2}, \quad K = \frac{1}{2}(M_0 + M_0^+) - \mathbb{1}\right)$$

$$M_\alpha = L_\alpha \sqrt{dt} \; , \quad \alpha = 1, 2, \dots$$

$$\mathbb{1} = \sum M_\alpha^+ M_\alpha = (\mathbb{1} + 2K)\, dt + \sum_{\alpha \geq 1} L_\alpha^+ L_\alpha \, dt + \cancel{O(dt^2)}$$

$$\Longrightarrow K = -\frac{1}{2} \sum_{\alpha \geq 1} L_\alpha^+ L_\alpha .$$

$$\Longrightarrow \rho(t + dt) = \underbrace{\left(\mathbb{1} + \left(\frac{-i}{\hbar} H + K\right) dt\right) \rho \left(\mathbb{1} + \left(\frac{i}{\hbar} H + K\right) dt\right)}_{= \rho + \left(-\frac{i}{\hbar}[H, \rho] + K\rho + \rho K\right) dt} + \sum L_\alpha \rho L_\alpha^+ dt$$

$$\Longrightarrow \boxed{\dot{\rho} = \mathcal{L}[\rho] = -\frac{i}{\hbar}[H, \rho] + \sum_{\alpha \geq 1} \left(L_\alpha \rho L_\alpha^+ - \frac{1}{2} L_\alpha^+ L_\alpha \rho - \frac{1}{2} \rho L_\alpha^+ L_\alpha\right)}$$

"Lindblad master equation"; $\mathcal{L}$: Liouville operator (Liouvillian)

Interpretation:

- $-\frac{i}{\hbar}[H, \rho]$: Hamiltonian evolution (but H can be renormalized due to interaction, e.g. Lamb shift)
- $L_\alpha \rho L_\alpha^+$ describes "measurement by environment", given by POVM $L_\alpha \sqrt{dt}$
- $-\frac{1}{2}(L_\alpha^+ L_\alpha \rho + \rho L_\alpha^+ L_\alpha)$ describes information acquired by environment by <u>absence</u> of measured event.

2) Examples

i) Master eq. for 2-level system coupled to vacuum

Consider 2-level atom cpl. to vacuum

Thought experiment: Monitor environment for photons.

For fixed (small) time interval dt, two possible outcomes:

$\alpha = 0$: no photon detected

$\alpha = 1$: photon detected

For $\alpha = 1$ (photon detected), system changes as

$$\rho_S \longrightarrow (L_1 \sqrt{dt})\, \rho_S\, (L_1 \sqrt{dt})^\dagger$$

What is the form of L_1? — L_1 describes what happens to system if photon is detected: Atom goes from $|e\rangle \rightarrow |g\rangle$.

$$\Rightarrow \quad L_1 = \sqrt{\Gamma}\, |g\rangle\langle e| = \sqrt{\Gamma}\, \sigma^-$$

(The rate Γ determines prob. of event $\alpha = 1$ in dt)

Event $\alpha = 0$ (no photon) is clearly the one with $M_0 = \mathbb{1} + O(dt)$.

System Hamiltonian:

$$H = \frac{\hbar \omega_{eg}}{2} \sigma_z \Rightarrow -\frac{i}{\hbar}[H, \rho_S] = -i \frac{\omega_{eg}}{2} [\sigma_z, \rho_S].$$

Master equation (with $\rho \equiv \rho_s$):

$$\frac{d\rho}{dt} = -i\,\frac{\omega_{eg}}{2}\left[\sigma_z, \rho\right] - \frac{\Gamma}{2}\left(\underbrace{\sigma^+\sigma^-}_{=|e\rangle\langle e|}\rho + \rho\sigma^+\sigma^- - 2\sigma^-\rho\sigma^+\right)$$

Differential eq. for matrix elements:

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} = \rho_{ee}|e\rangle\langle e| + \rho_{gg}|g\rangle\langle g| + \rho_{eg}|e\rangle\langle g| + \rho_{ge}|g\rangle\langle e|.$$

(with $\rho_{ge} = \rho_{eg}^*$):

$$[\sigma_z, \rho] = \sigma_z\rho - \rho\sigma_z = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ -\rho_{ge} & -\rho_{gg} \end{pmatrix} - \begin{pmatrix} \rho_{ee} & -\rho_{eg} \\ \rho_{ge} & -\rho_{gg} \end{pmatrix} = 2\begin{pmatrix} 0 & \rho_{eg} \\ -\rho_{ge} & 0 \end{pmatrix}$$

$$\sigma^+\sigma^-\rho = |e\rangle\langle e|\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ 0 & 0 \end{pmatrix}$$

$$\rho\sigma^+\sigma^- = \rho|e\rangle\langle e| = \begin{pmatrix} \rho_{ee} & 0 \\ \rho_{ge} & 0 \end{pmatrix}$$

$$\sigma^-\rho\sigma^+ = |g\rangle\langle e|\rho|e\rangle\langle g| = \begin{pmatrix} 0 & 0 \\ 0 & \rho_{ee} \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} \dot\rho_{ee} & \dot\rho_{eg} \\ \dot\rho_{ge} & \dot\rho_{gg} \end{pmatrix} = \begin{pmatrix} -\Gamma\rho_{ee} & -i\omega_{eg} - \frac{\Gamma}{2}\rho_{eg} \\ i\omega_{eg} - \frac{\Gamma}{2}\rho_{ge} & \Gamma\rho_{ee} \end{pmatrix}$$

Independent equations for diagonal & off-diagonal terms:

$$\dot{\rho}_{ee} = -\Gamma \rho_{ee} ; \quad \dot{\rho}_{gg} = \Gamma \rho_{ee}$$

$$\dot{\rho}_{eg} = -i\omega_{eg} - \frac{\Gamma}{2} \rho_{eg}$$

$\Rightarrow$ ρ_{ee} decays exponentially to zero (at rate Γ)

$\rho_{gg} = 1 - \rho_{ee}$ $\quad (\Leftrightarrow \mathrm{tr}\,\rho = 1)$

ρ_{eg} decays exp. to zero w/ oscillation