Detuned atom-cavity interaction: The dispersive regime

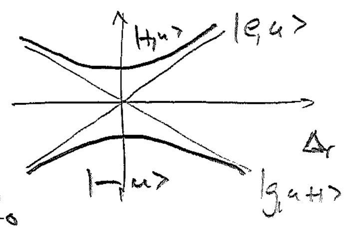
Reminder: Atom + cavity:

Eigenstates ("dressed states")

$$|+,n\rangle = \cos\frac{\theta_n}{2}|e,n\rangle + i\sin\frac{\theta_n}{2}|g,n+1\rangle$$

$$|-,n\rangle = \sin\frac{\theta_n}{2}|e,n\rangle - i\cos\frac{\theta_n}{2}|g,n+1\rangle$$

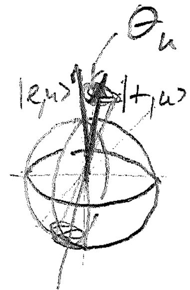
$$\tan\theta_n = \frac{\Omega_n}{\Delta_r}\,;\ \theta_n\in[0;\pi]$$



Energies: $E_n^{\pm} = \hbar\omega_r(n+1) \pm \frac{\hbar}{2}\sqrt{\Delta_r^2+\Omega_n^2}\,,\quad \Omega_n = \sqrt{n+1}\,\Omega_0$

Large detuning $|\Delta_r| \gg \Omega_n$ ($\to$ dep. on photon #!)

	$\Delta_r>0$:	$\Delta_r<0$:
Eigenstates:	$|+,n\rangle \approx |e,n\rangle$	$|+,n\rangle \approx |g,n\rangle$
	$|-,n\rangle \approx |g,n\rangle$	$|-,n\rangle \approx |e,n\rangle$



$\Rightarrow$ Basis states $|e,n\rangle$ & $|g,n+1\rangle$ almost don't change!

Energies

$$E_n^{\pm} = \hbar\omega_r(n+1) \pm \frac{\hbar}{2}|\Delta_r|\sqrt{1+\frac{\Omega_n^2}{\Delta_r^2}}$$

$$\approx \hbar\omega_r(n+1) \pm \frac{\hbar}{2}\left(|\Delta_r| + \frac{\Omega_n^2}{2|\Delta_r|}\right)$$

$\Rightarrow$ Energy of $|e,n\rangle$:

$$\left.\begin{array}{l}\Delta_r>0: |e,n\rangle \approx |+,n\rangle \\ \Delta_r<0: |e,n\rangle \approx |-,n\rangle\end{array}\right\} \Rightarrow E_{e,n} = \hbar\omega_r(n+1) + \frac{\hbar}{2}\Delta_r + \frac{\hbar\overbrace{\Omega_n^2}^{\Omega_0^2(n+1)}}{4\Delta_r}$$

$$= \hbar\omega_r\left(n+\tfrac{1}{2}\right) + \frac{\hbar}{2}\omega_{eg} + \frac{\hbar\Omega_0^2}{4\Delta_r}(n+1)$$

Similarly: $E_{g,n+1} = \hbar\omega_r\left(n+\frac{3}{2}\right) - \frac{\hbar}{2}\omega_{eg} - \frac{\hbar\Omega_0^2}{4\Delta_r}(n+1)$

$$\Rightarrow \quad H_{eff} = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right) - \frac{\hbar}{2}\omega_{eg}\sigma_z + H_{disp}$$

with $$\underline{\underline{H_{disp}}} = \begin{cases} \hbar s_0\cdot(n+1), & \text{atom in } |e\rangle \\ -\hbar s_0\cdot n, & \text{atom in } |g\rangle \end{cases}, \quad \text{with } s_0 = \frac{\Omega_0^2}{4\Delta_r}.$$

$$\underline{\underline{= \hbar\left(a^\dagger a\cdot\sigma_z + \sigma^+\sigma^-\right) s_0}}\;.$$

Consequences:

*) Lamb shift: Atomic transition freq. ω_{eg} changed by
$\delta\omega_{eg} = (2n+1)s_0$.

- $\underline{\underline{2ns_0}}$: level shift induced by light field in cavity
  $\Rightarrow$ has classical analogue!
- $\underline{\underline{1\times s_0}}$: shift of atomic levels in <u>empty cavity</u> induced by vacuum fluctuations: <u>Lamb shift</u> in cavity.
  (Normal Lamb shift: cpl. to continuum of modes in free space)

* Change of cavity frequency:
opposite of Lamb shift: atom in $|e\rangle$ enters cavity
$\Rightarrow$ energy shift $(2n+1)s_0$ $\Rightarrow$ cavity freq. ($\equiv$ energy/photon) shifted by $\delta\omega_e = s_0$; vs. atom in $|g\rangle$: $\delta\omega_g = -s_0$.
$\Rightarrow$ atom is like piece of dielectric w/ (freq.-dependent $\equiv$ dispersive) refractive index changing cavity freq.
$\Rightarrow$ can be used to detect state of atom by meas. cavity resonance frequency.

* mechanical effect:

Atom in $|e\rangle$ enters cavity w/ n photons; $\Delta > 0$:

$\Rightarrow$ energy increases by $(n+1)\hbar S_0$!

Where does energy come from?

$\Rightarrow$ Repulsive force when atom enters cavity!

Atom in $|g\rangle$, $\Delta > 0 \Rightarrow$ energy decreases $\Rightarrow$ attractive force!

($\Delta < 0$: opposite effect!)

* measurement of one subsystem:

$$H_{eff} = \hbar\omega_c\left(a^\dagger a + \tfrac{1}{2}\right) + \frac{\hbar}{2}\left(\sigma_z \omega_{eg}\right) + \hbar S_0 \overbrace{\left(\sigma_z a^\dagger a + \sigma_+ \sigma_-\right)}^{\equiv \sigma_z\left(a^\dagger a + \frac{1}{2}\right) + \frac{1}{2}}$$

$$\overset{(1)}{=} \frac{\hbar}{2}\sigma_z\left(\omega_{eg} + S_0(2a^\dagger a + 1)\right) + \text{atom-indep. terms}$$

$$\overset{(2)}{=} \hbar a^\dagger a\left(\omega_c + S_0 \sigma_z\right) + \text{field-indep. terms}$$

Note: All terms in H_{eff} preserve states in $\{|g,n\rangle, |e,n\rangle\}$ basis!

(1) $\Rightarrow$ describes effective z-field on atom w/ strength $\omega_{eg} + a^\dagger a + \frac{1}{2}$

prepare atom in $\frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$

$\Rightarrow$ interaction induces phase shift $\frac{1}{\sqrt{2}}\left(e^{-i\phi/2}|e\rangle + e^{-i\phi/2}|g\rangle\right)$

with $\phi = \left(\frac{\omega_{eg}}{2} + a^\dagger a + \frac{1}{2}\right)t$

$\Rightarrow$ meas. of atom in XY-plane (possibly repeated w/ many atoms) reveals ϕ and thus # of photons.

(2) $\Rightarrow$ similarly: allows to meas. state of atom via state of light field.

* Entangling light & matter:

Cavity initially in $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$; atom in $\frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$:

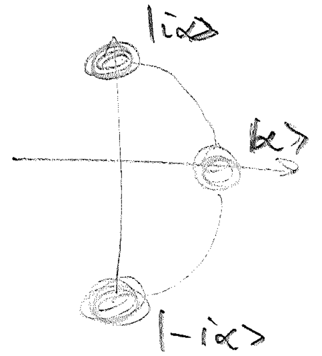
Interaction picture: $H_{int} = \hbar s_0 \, a^\dagger a \cdot \sigma_z$

$$|e\rangle|\alpha\rangle \longrightarrow |e\rangle|\alpha e^{i\phi}\rangle$$

$$|g\rangle|\alpha\rangle \longrightarrow |g\rangle|\alpha e^{-i\phi}\rangle \quad ; \; \phi = s_0 t$$

Choose time s.th. $\phi = \pi/2$:

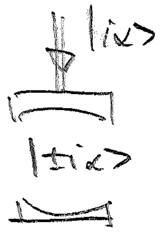
$$|\psi(t)\rangle = |e\rangle|i\alpha\rangle + |g\rangle|-i\alpha\rangle$$



$\alpha \gg 1$: $|i\alpha\rangle$ and $|-i\alpha\rangle$ essentially orthogonal & "classical"

$\Rightarrow$ "Schrödinger cat state" (macroscopic quantum superposition)

Note: State of "cat" in cavity can be measured experimentally by mixing a coherent pulse w/ $|i\alpha\rangle$ into cavity:



$$\Rightarrow \left.\begin{array}{l} |i\alpha\rangle \longrightarrow |2i\alpha\rangle \\ |-i\alpha\rangle \longrightarrow |0\rangle \end{array}\right\}$$ can be distinguished by e.g. sending atom into cavity.

IV. Interaction with the environment

1) Formalism: mixed states + Lindblad master equation

i) Mixed states:

* pure state $|\psi\rangle$, observable O:

  expectation value $\langle\psi|O|\psi\rangle = \mathrm{tr}\left[O|\psi\rangle\langle\psi|\right]$

  (where $\mathrm{tr}(X) = \sum_i \langle\phi_i|X|\phi_i\rangle$ for any ONB $|\phi_i\rangle$)

* ensemble: state $|\psi_i\rangle$ with prob. p_i:

  $$\text{exp. value } \sum p_i \langle\psi_i|O|\psi_i\rangle = \sum p_i \mathrm{tr}\left[O|\psi_i\rangle\langle\psi_i|\right]$$
  $$= \mathrm{tr}\left[O\left(\sum p_i|\psi_i\rangle\langle\psi_i|\right)\right] = \mathrm{tr}[O\rho]$$

  with $\rho = \sum p_i|\psi_i\rangle\langle\psi_i|$ the density operator (density matrix, mixed state)

  It holds that $\rho \geq 0$; $\mathrm{tr}(\rho) = 1$.

* consider bipartite state

  $$|\psi\rangle = \sum w_{ij}|\phi_i\rangle_S|\chi_j\rangle_E \quad (S = \text{system}, E = \text{environment})$$

$\Rightarrow$ exp. value of observable O_S on S is:

$$\langle\psi|O_S\otimes \mathbb{1}_E|\psi\rangle = \sum w_{ij} w^*_{i'j'} \langle\phi_{i'}|O_S|\phi_i\rangle_S \underbrace{\langle\chi_{j'}|\mathbb{1}_E|\chi_j\rangle}_{=\delta_{jj'}}$$
$$= \mathrm{tr}[O_S\rho_S]$$

with $\rho_S = \sum w_{ij} w^*_{i'j} |\phi_i\rangle\langle\phi_{i'}| = \mathrm{tr}_E\left[|\psi\rangle\langle\psi|\right]$

(with "partial trace" $\mathrm{tr}_E X = \sum(\mathbb{1}_S\otimes\langle\mu_j|_E)X(\mathbb{1}_S\otimes|\mu_j\rangle_E$, $|\mu_j\rangle$ ONB)

<u>Conversely:</u> Any mixed state ρ ($\rho \geq 0$, $\mathrm{tr}\,\rho = 1$) can be interpreted as part of a pure state ("purification") on larger system:

- $\rho_S = \sum p_i |\phi_i\rangle_S\langle\phi_i|$; $|\phi_i\rangle$ ONB
- Pick arbitrary ONS $|\chi_i\rangle$ on E.
- $|\psi\rangle = \sum \sqrt{p_i}\, |\phi_i\rangle_S |\chi_i\rangle_E \Rightarrow \mathrm{tr}_E |\psi\rangle\langle\psi| = \rho_S$

$\Rightarrow |\psi\rangle$ is <u>purification of ρ_S</u>.

<u>Note:</u> Any two purifications are related by a basis trafo on E.