Last lecture: Interaction of two-level system w/ class. light:

$$H = \underbrace{\frac{\hbar\omega_{eg}}{2}\sigma_z}_{\text{two-level atom}} - \underbrace{d\left(\underline{\varepsilon}_a\sigma^- + \underline{\varepsilon}_a^*\sigma^+\right)}_{\text{dipole moment of atomic transition}} \cdot \underbrace{i\,\mathcal{E}_r\left(\underline{\varepsilon}_r e^{-i\omega_r t} - \underline{\varepsilon}_r^* e^{i\omega_r t}\right)}_{\text{classical EM field}}$$

Write $\frac{\hbar\omega_{eg}}{2}\sigma_z = \underbrace{\frac{\hbar\omega_r}{2}\sigma_z}_{\text{energy of photon}} + \frac{\hbar\Delta_r}{2}\sigma_z$; $\underbrace{\Delta_r = \omega_{eg} - \omega_r}_{\text{"detuning"}}$

Interaction picture w.r.t. $H_0 = \frac{\hbar\omega_r}{2}\sigma_z$:

$$H_I(t) = e^{iH_0 t}\left(H - H_0\right)e^{-iH_0 t}$$

State evolves as $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H_I(t)|\psi\rangle$

Observables: $O_I(t) = e^{iH_0 t}\,O\,e^{-iH_0 t}$

$\Rightarrow \sigma_z(t) = \sigma_z(0) \Rightarrow$ meas. of state of atom ($|g\rangle$ vs. $|e\rangle$) unaffected by trafo!

We find

$$H_I(t) = \frac{\hbar\Delta_r}{2}\sigma_z - i\,\mathcal{E}_r d\left(\underline{\varepsilon}_a e^{-i\omega_r t}\sigma^- + \underline{\varepsilon}_a^* e^{i\omega_r t}\sigma^+\right)\cdot\left(\underline{\varepsilon}_r e^{-i\omega_r t} - \underline{\varepsilon}_r^* e^{i\omega_r t}\right)$$

$$= \frac{\hbar\Delta_r}{2}\sigma_z - i\,\mathcal{E}_r d\left(-\underline{\varepsilon}_a\cdot\underline{\varepsilon}_r^*\sigma^- + \underline{\varepsilon}_a^*\cdot\underline{\varepsilon}_r\sigma^+ + \underline{\varepsilon}_a\cdot\underline{\varepsilon}_r\sigma^- e^{-2i\omega_r t} + \underline{\varepsilon}_a^*\cdot\underline{\varepsilon}_r^*\sigma^+ e^{2i\omega_r t}\right)$$

Rotating wave approximation: Neglect rapidly oscillating terms $e^{\pm 2i\omega_r t}$ $(\omega_r \gg \Delta_r)$ (these average out on very short time scales!)

$$\Rightarrow H_I = \frac{\hbar \Delta_r}{2}\sigma_z - i\hbar\frac{\Omega_r}{2}\left(e^{-i\varphi}\sigma^+ - e^{i\varphi}\sigma^-\right)$$

with $\Omega_r = \frac{2d}{\hbar}\mathcal{E}_r \underbrace{\varepsilon_a^* \cdot \varepsilon_r e^{i\varphi}}_{\varphi \text{ s.th. } > 0}$ the "(classical) Rabi frequency"

In spin/qubit notation:

$$H_I = \frac{\hbar}{2}\begin{pmatrix} \Delta_r & -i\Omega_r e^{-i\varphi} \\ i\Omega_r e^{i\varphi} & -\Delta_r \end{pmatrix} = \frac{\hbar}{2}\Omega_r' \, \vec{\sigma}\cdot\vec{u}$$

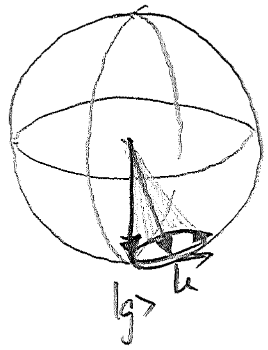
with $\Omega_r' = \sqrt{\Delta_r^2 + \Omega_r^2}$; $\vec{u} = \frac{1}{\Omega_r'}\begin{pmatrix} -\Omega_r \sin\varphi \\ -\Omega_r \cos\varphi \\ \Delta_r \end{pmatrix}$.

$\Rightarrow$ H_I describes spin in magnetic field of strength Ω_r' oriented along $\vec{u}$!

$\Rightarrow$ Spin precesses around $-\vec{u}$ at frequency Ω_r'!
(cf. Homework Sheet 5).
(It behaves just as a class. spin!)

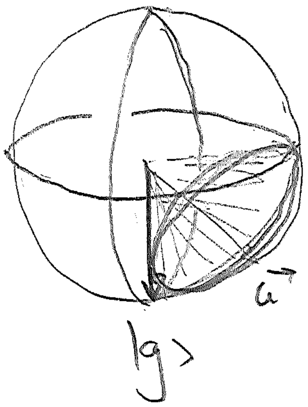
Different cases of detuning Δ_r – atom initially in $|g\rangle$:

1) Large detuning $\Delta_r \gg \Omega_r$: $\vec{u} \approx \underline{e}_z$



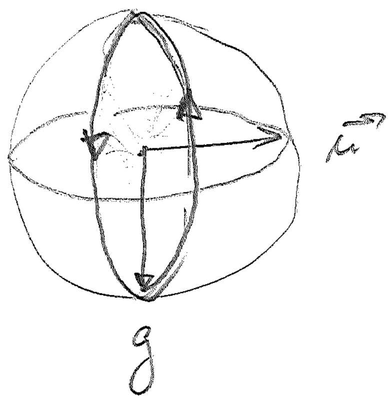
precession on small cone
→ state close to $|g\rangle$ at all times

2) $\Delta_r \approx \Omega_r$:



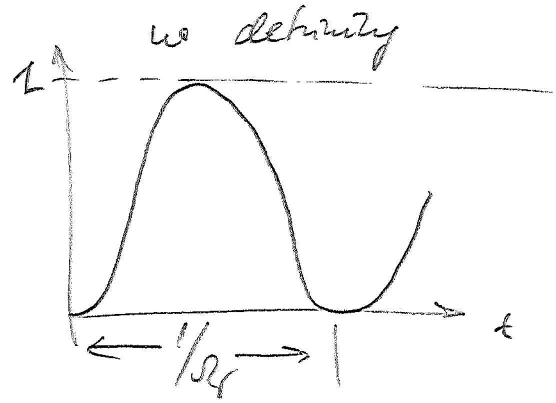
precession betw. $|g\rangle$ and $\frac{1}{\sqrt{2}}(|g\rangle + |e\rangle)$

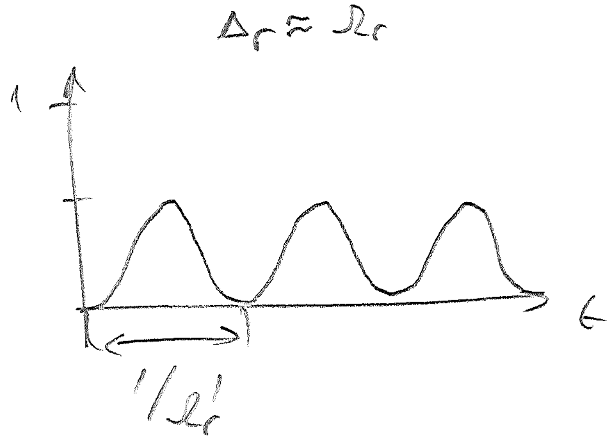
3) small detuning $\Delta_r \ll \Omega_r$: $\vec{u} \approx \underline{e}_x$ (for $\varphi = \pi/2$)

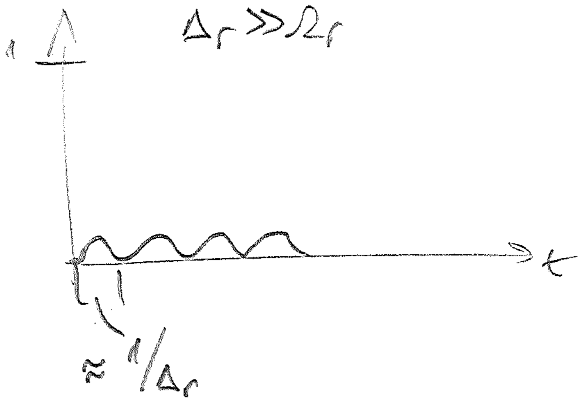


state precesses on circle
between $|g\rangle$ and $|e\rangle$.

Measurement of prob. of state $|e\rangle$ (or population for an ensemble) as a function of time:







"Rabi oscillations"
(Note: $\Omega_r \propto |\mathcal{E}_r|$ = amplitude of EM field)

Controlled Rabi pulses can be used to prepare atoms in desired state, e.g. $|e\rangle$, or $\frac{1}{\sqrt{2}}\left(|e\rangle + |g\rangle\right)$, and perform controlled interference experiments $\longrightarrow$

$\longrightarrow$ Ramsey spectroscopy.

Interaction of QM light w/ two-level systems:

Replace class. light by QM light field (single mode):

$$H = \underbrace{\frac{\hbar\omega_{eg}}{2}\sigma_z}_{\text{atom}} + \underbrace{\hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right)}_{\text{light}} \underbrace{- d\left(\varepsilon_a\sigma^- + \varepsilon_a^*\sigma^+\right)}_{\text{atomic dipole moment}} \cdot \underbrace{i\,\mathcal{E}_0\left(\varepsilon_r a - \varepsilon_r^* a^\dagger\right)}_{\text{QM EM field}}$$

Rotating wave approximation:

If $\omega_{eg} \approx \omega_r$, the terms $\sigma^- a$ and $\sigma^+ a^\dagger$ are highly non-resonant (in interaction picture, they rotate at $\approx 2\omega_r$, i.e., they average out on very short time scales) $\rightarrow$ neglect! (RWA)

$$\Rightarrow H = \hbar\omega_r\left(\frac{\sigma_z}{2} + a^\dagger a + \frac{1}{2}\right) + \frac{\hbar\Delta_r}{2}\sigma_z - \frac{i\hbar\Omega_0}{2}\left(\sigma^+ a - \sigma^- a^\dagger\right),$$

with $\Omega_0 = \frac{2d\,\mathcal{E}_0}{\hbar}\,\varepsilon_a^*\cdot\varepsilon_c$ the vacuum Rabi frequency

(wlog: $\varepsilon_a^*\cdot\varepsilon_c > 0$!)

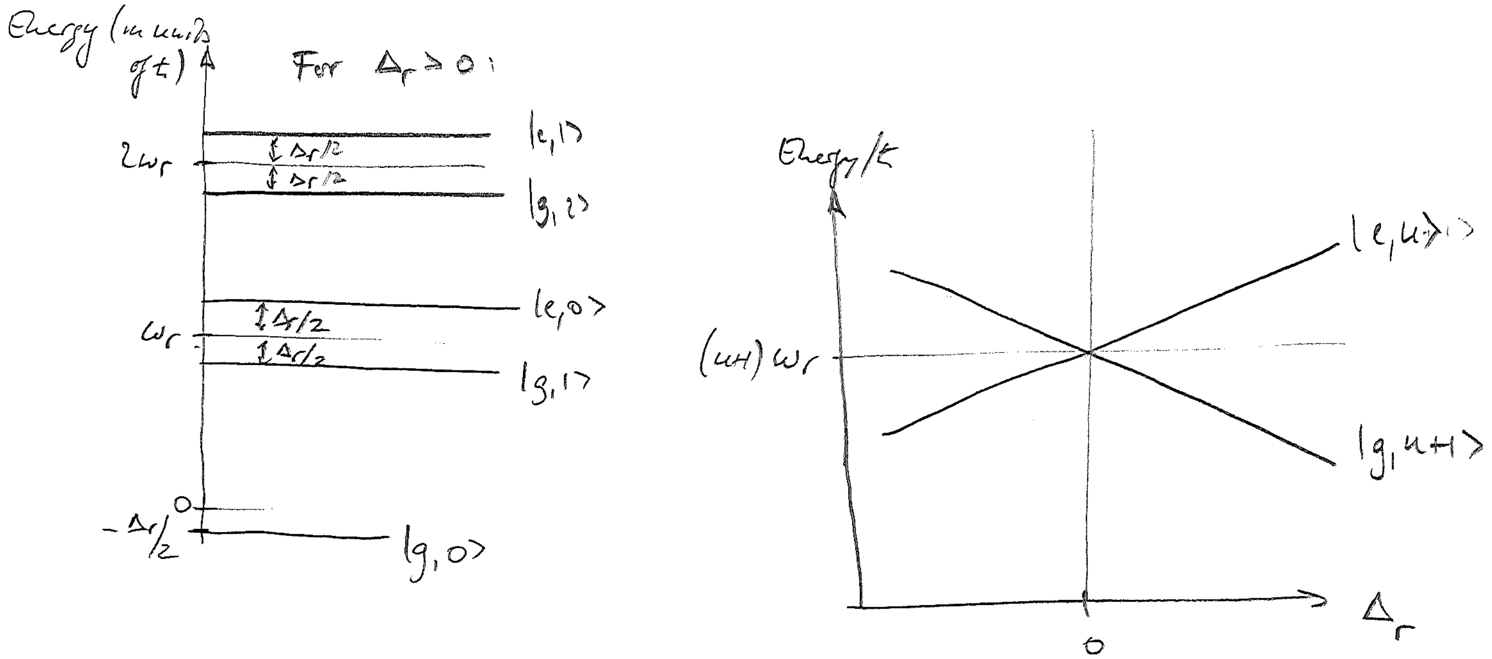
"Jaynes-Cummings model"

Let us first consider the uncoupled system,

$$H_0 = \hbar\omega_r\left(\frac{\sigma_z}{2} + a^\dagger a + \frac{1}{2}\right) + \frac{\hbar\Delta_r}{2}\sigma_z \;;\; \Delta_r \ll \omega_r.$$

The eigenstates $|e,n\rangle = |e\rangle_{at}|n\rangle_{light}$ and $|g,n+1\rangle = |g\rangle_{atom}|n+1\rangle_{light}$

are approx. degenerate, w/ splitting $\hbar\Delta_r$.



Interaction term $H_{int} = -i\hbar\Omega_0/2\,(\sigma^+ a - \sigma^- a^\dagger)$

– couples only states $|e,n\rangle \longleftrightarrow |g,n+1\rangle$

$\Rightarrow H = H_0 + V$ can be diagonalized in two-dim. subspaces spanned by $\{|e,n\rangle, |g,n+1\rangle\}$.

$$H_{0,n} = \left[\hbar\omega_r(n+1) + \frac{\Delta_r}{2}\right]|e,n\rangle\langle e,n| + \left[\hbar\omega_r(n+1) - \frac{\Delta_r}{2}\right]|g,n+1\rangle\langle g,n+1|$$

$$= \hbar\omega_r(n+1)\,\mathbb{1} + \frac{\hbar}{2}\Delta_r\sigma_z$$

$$H_{int,n} = \frac{-i\hbar\Omega_0}{2}\left(\sqrt{n+1}\,|e,n\rangle\langle g,n+1| - \sqrt{n+1}\,|g,n+1\rangle\langle e,n|\right)$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & -i\Omega_n \\ i\Omega_n & 0 \end{pmatrix} = \frac{\hbar}{2}\,\Omega_n\,\sigma_y$$

$$\underbrace{}_{|e,n\rangle} \quad \underbrace{}_{|g,n+1\rangle}$$

with $\Omega_n = \sqrt{n+1}\,\Omega_0$ the "n-photon Rabi frequency"

$$\Rightarrow \quad H_n = \hbar\omega_r(n+1)\,\mathbb{1} + V_n, \quad \text{with}$$

$$\underline{V_n = \frac{\hbar}{2}\begin{pmatrix} \Delta_r & -i\Omega_n \\ i\Omega_n & -\Delta_r \end{pmatrix} = \frac{\hbar}{2}\left(\Delta_r\sigma_z + \Omega_n\sigma_y\right).}$$

$\rightarrow$ Formally analogous to spin in magnetic field!

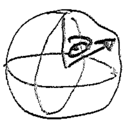
<u>Diagonalization of V_n:</u>

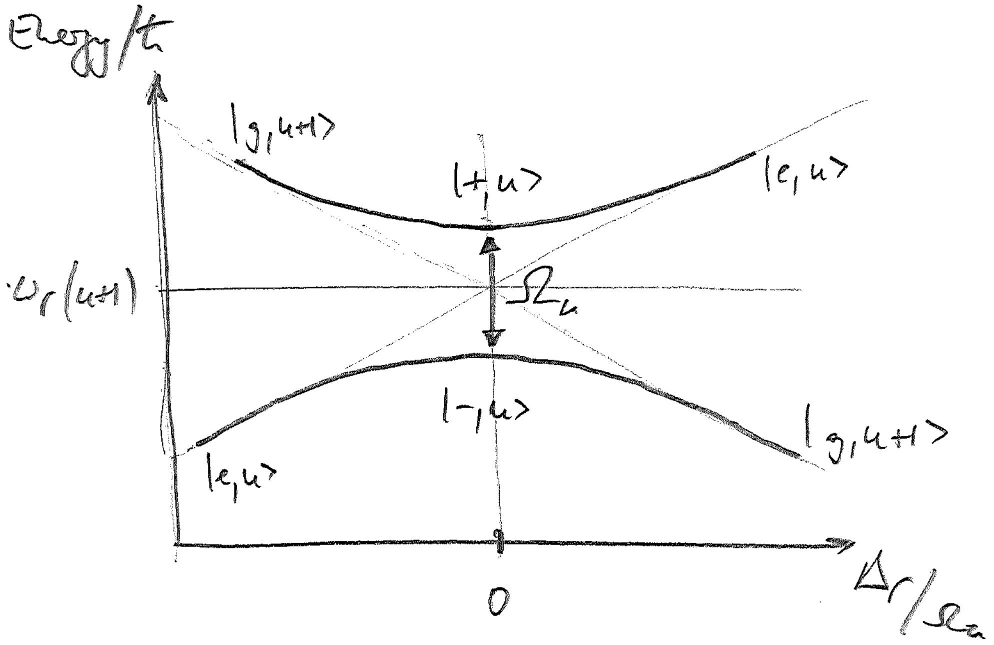
$$V_n = \underbrace{\frac{\hbar}{2}\sqrt{\Delta_r^2+\Omega_n^2}}_{\text{eigenvalues}}\;(\vec{n}\cdot\vec{\sigma}), \quad \text{with } \vec{n} = \frac{1}{\sqrt{\Delta_r^2+\Omega_n^2}}\begin{pmatrix} 0 \\ \Omega_n \\ \Delta_r \end{pmatrix}.$$

<u>Eigenvalues</u> of H_n: $E_n^{\pm} = \hbar\omega_r(n+1) \pm \frac{\hbar}{2}\sqrt{\Delta_r^2+\Omega_n^2}$

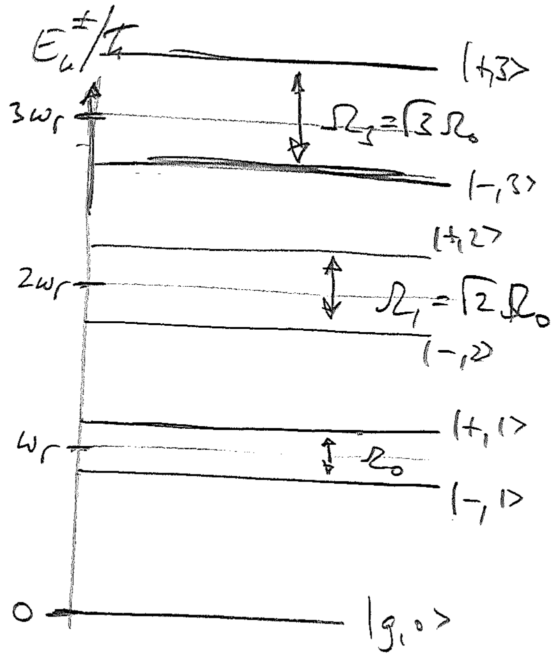
<u>Eigenstates:</u>

$$\left.\begin{aligned} |+,n\rangle &= \cos\tfrac{\theta_n}{2}\,|e,n\rangle + i\sin\tfrac{\theta_n}{2}\,|g,n+1\rangle \\ |-,n\rangle &= \sin\tfrac{\theta_n}{2}\,|e,n\rangle - i\sin\tfrac{\theta_n}{2}\,|g,n+1\rangle \end{aligned}\right\} \boxed{\text{"dressed states"}}$$

(with $\tan\theta_n = \Omega_n/\Delta_r$, angle on Bloch sphere: )



<u>Resonant case:</u> $\Delta_r = 0 \quad (\Rightarrow \theta_n = \pi/2)$



$$E_n^{\pm} = \hbar\omega_r(n+1) \pm \frac{\hbar}{2}\Omega_n$$

$$|\pm,n\rangle = \frac{1}{\sqrt{2}}\left(|e,n\rangle \pm i\,|g,n+1\rangle\right)$$

Prepare atom in $|e\rangle$ and place it in field (e.g. cavity) containing n photons:

$$|\psi(t=0)\rangle = |e,n\rangle = \frac{1}{\sqrt{2}}\left(|+,n\rangle + |-,n\rangle\right)$$

Time evolution under H_n (neglecting const. term $\hbar\omega_r(n+\frac{1}{2})$):

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\Omega_n t/2}\,|+,n\rangle + e^{i\Omega_n t/2}\,|-,n\rangle\right)$$

$$= \cos\left(\frac{\Omega_n t}{2}\right)|e,n\rangle + \sin\left(\frac{\Omega_n t}{2}\right)|g,n+1\rangle$$

Similarly, if atom initially prepared in $|g\rangle$ & $|n+1\rangle$ photons in cavity:

$$|\psi(t)\rangle = -\sin\left(\frac{\Omega_n t}{2}\right)|e,n\rangle + \cos\left(\frac{\Omega_n t}{2}\right)|g, n+1\rangle .$$

(Note: If atom in $|g\rangle$ & cavity empty: $|n=0\rangle$, nothing happens.)

System exchanges energy betw. atom & EM field at frequency Ω_n $\Longrightarrow$ Quantum Rabi oscillations.

Distinct quantum features as compared to class. Rabi oscillations:

- Rabi freq. $\Omega_n \propto \sqrt{n+1}$ dep. on # photons in cavity
- Excited atom in empty cavity / vacuum:

  $$\Omega_0 = \frac{2d\mathcal{E}_0}{\hbar} \neq 0$$

  $\Longrightarrow$ atom emits photon into empty cavity (vacuum Rabi oscillation).

  w/out cavity $\Longrightarrow$ spontaneous decay of excited atom in vacuum!

  (cf. class. light: $\Omega_R \propto |E| \Rightarrow$ zero in vacuum)