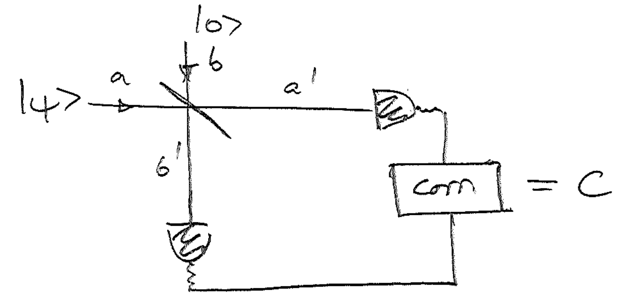
Coherence properties of the EM field



$$C = \langle a'^{\dagger} b'^{\dagger} b' a' \rangle = \frac{1}{4}\langle \psi,0|(a+b)^{\dagger}(a-b)^{\dagger}(a-b)(a+b)|\psi,0\rangle$$

$$= \frac{1}{4}\langle \psi | a^{\dagger}a^{\dagger}aa|\psi\rangle = \frac{1}{4}\left[\left(\Delta n_{in}^2 + \bar{n}_{in}^2\right) - \bar{n}_{in}\right]$$

More generally: Source $|\psi\rangle$ varies slowly in time, and path b' is by $c\tau$ longer than path a:

$$C = \langle a^{\dagger}(t)\, a^{\dagger}(t+\tau)\, a(t+\tau)\, a(t)\rangle$$

If system is stationary (i.e., C indep. of t), this is the second order coherence function

$$G^{(2)}(\tau) := \langle a^{\dagger}(0)\, a^{\dagger}(\tau)\, a(\tau)\, a(0)\rangle .$$

Normalized version: $$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{\left(\langle a^{\dagger}(0)a(0)\rangle\right)^2}$$

$g^{(2)}(\tau)$ measures if photons are more likely to arrive together (bunching, $g^{(2)}(0) > g^{(2)}(\tau)$) or separated (anti-bunching, $g^{2}(0) < g^{2}(\tau)$).

We assume that for $\tau \to \infty$, all correlations vanish $\Rightarrow g^{(2)}(\tau) \to 1$.

i.e.: bunching $\Longleftrightarrow g^{(2)}(0) > 1$

anti-bunching $\Leftrightarrow g^{(2)}(0) < 1$.

What do we expect classically? ($\langle\cdot\rangle \hat{=}$ time-average)

$$\langle |E(0)|^2 |E(\tau)|^2 \rangle \underset{\substack{\uparrow \\ \text{Cauchy-Schwarz}}}{\leq} \sqrt{\langle |E(0)|^4 \rangle} \cdot \sqrt{\underbrace{\langle |E(\tau)|^4 \rangle}_{= \langle |E(0)|^4 \rangle}}$$

$$\Longrightarrow \underline{\underline{g^{(2)}(\tau) \leq g^{(2)}(0).}}$$

$\Longrightarrow$ Classical light always exhibits bunching!

Quantum mechanical light:

$$g^{(2)}(0) = \frac{\langle a^+ a^+ a a \rangle}{(\langle a^+ a \rangle)^2} = \frac{(\Delta n)^2 + \bar{n}^2 - \bar{n}}{\bar{n}^2} = 1 + \frac{(\Delta n)^2 - \bar{n}}{\bar{n}^2}$$

$\Rightarrow$ $\Delta n > \sqrt{\bar{n}}$ (super-Poissonian statistics) $\Rightarrow$ bunching

$\Delta n < \sqrt{\bar{n}}$ (sub-Poissonian stat.) $\Longrightarrow$ anti-bunching

Coherent state: $\Delta n = \sqrt{\bar{n}} \Rightarrow g^{(2)}(0) = 1$

$\rightarrow$ on the edge of class. & non-class.

Fock state: $\Delta n = 0 \Rightarrow g^{(2)}(0) = 1 - \frac{1}{n} \Rightarrow$ anti-bunching

Squeezed light: can exhibit anti-bunching dep. on α, r.

III. Interaction of light and matter

1. Interaction of classical light with one atom

Consider single mode class. EM field, and one atom with one valence electron (= alkali atom – other electrons don't contribute to low-energy physics):

Hamiltonian:

$$H = \frac{1}{2m}\left(\hat{p} - q\underline{A}(\underline{r},t)\right) + q\mathcal{U}(\underline{r},t) \qquad \begin{pmatrix} \hat{p} = -i\hbar\nabla \\ q = -e \end{pmatrix}$$

Coulomb gauge for EM field ($\nabla\cdot\underline{A} = 0$)

$$\underline{A}(\underline{r},t) = \frac{E_0}{\omega}\cos(\omega t - \underline{k}\cdot\underline{r}) \quad ; \quad \mathcal{U}_{EM}(\underline{r},t) = 0$$

$$\left(\text{and } \underline{B} = \nabla\times\underline{A}, \quad E = -\frac{\partial}{\partial t}A - \underbrace{\nabla\mathcal{U}}_{=0}\right)$$

Atomic Hamiltonian:

$$H = \frac{1}{2m}\hat{p}^2 + q\,\mathcal{U}_{atom}(\underline{r})$$

Total Ham.:

$$\Rightarrow \quad H = \frac{1}{2m}\left(\hat{p} - eA(\underline{r},t)\right)^2 + q\,\mathcal{U}_{atom}(\underline{r},t)$$

Coulomb Gauge $\Rightarrow$ $\hat{p}\cdot\underline{A} = \underline{A}\cdot\hat{p}$

$$\Rightarrow \quad H = \underbrace{\frac{1}{2m}\hat{p}^2 + q\,\mathcal{U}_{atom}(\hat{r},t)}_{\equiv H_{atom}} \underbrace{- \frac{q}{m}\hat{p}\cdot\underline{A}(\underline{r},t)}_{\equiv H_{I,1}} + \underbrace{\frac{q^2\underline{A}^2(\underline{r},t)}{2m}}_{\equiv H_{I,2}}$$

$\Rightarrow \quad H = H_{atom} + H_{I,1} + H_{I,2}$

Long-wavelength approximation:

wavelength of light $\sim 10^{-6}\,m$

size of atom $\sim 10^{-10}\,m$

$\Rightarrow$ field approx. constant over size of atom

$\Rightarrow$ replace $\vec{r}$ by $\vec{r}_0$ (= center of atom)

$\Rightarrow$ $H_{I,2}$ is $\vec{r}$-indep. Scalar $\Rightarrow$ irrelevant (only energy shift)

$\Rightarrow$ light-atom interaction described by $H_{I,1} = -\frac{q}{m}\hat{p}\cdot\underline{A}(\underline{r}_0, t)$

("A·p Hamiltonian")

Different rep. of interaction:

Choose Goeppert-Mayer gauge:

$$\underline{A}' = \underline{A} + \nabla F$$
$$u' = u - \frac{\partial}{\partial t}F$$

with $F = -(\underline{r}-\underline{r}_0)\cdot\underline{A}(\underline{r}_0, t)$

$\Rightarrow \quad \underline{A}'(\underline{r}, t) = \underline{A}(\underline{r}, t) - A(r_0, t) \approx 0$ (long-WL approx.)

$$u'(\underline{r}, t) = u_{atom}(\underline{r}) + (r - r_0)\underbrace{\frac{\partial}{\partial t}\underline{A}(\underline{r}_0, t)}_{= -\underline{E}(r_0, t)}$$

$= -\underline{E}(r_0 t)$: field of EM wave

$$\Rightarrow H = H_{atom} + H_I$$

$$H_{atom} = \frac{p^2}{2} + q\,U_{atom}, \quad \text{and}$$

$$H_I = \underbrace{-q(r-r_0)}_{\equiv \hat{D}:\ \text{dipole moment operator}} E(r_0,t)$$

"electric dipole Hamiltonian"

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2. Two-level atoms:

Focus first on "atoms with only two levels" $|g\rangle$ and $|e\rangle$, with energies $E_e > E_g$. This is a good approx. if $E_e - E_g \approx \hbar\omega$ (light freq.) while all other transitions are far away (or otherwise ruled out, e.g. symmetries, metastable state, ...).

Two-level atom can be modelled as pseudo-spin-$\frac{1}{2}$, or qubit:

$$|g\rangle \equiv |\downarrow\rangle \equiv |1\rangle \qquad (!)$$
$$|e\rangle \equiv |\uparrow\rangle \equiv |0\rangle$$

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |g\rangle\langle e| + |e\rangle\langle g|$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i|g\rangle\langle e| - i|e\rangle\langle g|$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |e\rangle\langle e| - |g\rangle\langle g|$$

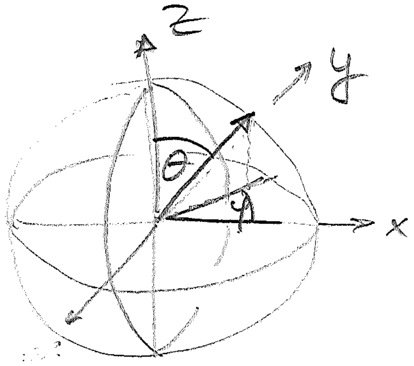
$$\sigma^+ = \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |e\rangle\langle g|; \quad \sigma^- = \frac{\sigma_x - i\sigma_y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |g\rangle\langle e|$$

$$H = \frac{\hbar \omega_{eg}}{2} \sigma_z , \quad \text{with} \quad \hbar \omega_{eg} = E_e - E_g$$

(up to additive constant)

Bloch sphere picture:

Every pure qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is (up to a phase) of the form $|\psi\rangle \propto \cos\theta/_2 \, |0\rangle + e^{i\phi} \sin\theta/_2 \, |1\rangle$:



qubit state can be pictured as vector on unit sphere (Bloch sphere)

$\longleftrightarrow$ classical spin picture.

Alternative derivation: $|\psi\rangle\langle\psi| = \frac{1}{2}\mathbb{1} + \frac{1}{2}\vec{v}\cdot\vec{\sigma}$, (Note: $\vec{v}\cdot\vec{\sigma} = v_1\sigma_x + v_2\sigma_y + v_3\sigma_z$)

where $\vec{v}$, $|\vec{v}| = 1$ is a vector on the unit circle,

$$\vec{v} = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix} \longrightarrow \text{Homework}$$

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3. Two-level atom in class. EM field – Rabi oscillations:

$$H_{atom} = \frac{\hbar\omega_{eg}}{2}\sigma_z$$

$$H_I = -\hat{\underline{D}} \cdot \underline{E}$$

Matrix elements of $\hat{\underline{D}}$ in $|g\rangle, |e\rangle$:

$$\langle g|\hat{\underline{D}}|g\rangle = \int \underbrace{|\psi_g(\underline{r})|^2}_{\text{even}} \, q \underbrace{\underline{r}}_{\text{odd}} \, d\underline{r} = 0$$

$$\langle e|\hat{\underline{D}}|e\rangle = 0$$

$$\langle g|\hat{\underline{D}}|e\rangle =: d\,\underline{\varepsilon}_a$$

d ← dipole matrix element (real)

$\underline{\varepsilon}_a$ ← polarization (complex) – dep. on angular momentum of $|e\rangle, |g\rangle$.

$$\Rightarrow H_I = -d\left(\underline{\varepsilon}_a \sigma^- + \underline{\varepsilon}_a^* \sigma^+\right)\cdot \underline{E}$$

with $\underline{E} = i\,\mathcal{E}_r\left[\underline{\varepsilon}_r e^{-i\omega_r t} - \underline{\varepsilon}_r^* e^{i\omega_r t}\right]$

$\underline{\varepsilon}_r$: polarization (cplx.) of light; $\mathcal{E}_r > 0$.

$$\Rightarrow H = \frac{\hbar\omega_{eg}}{2}\sigma_z - d\left(\underline{\varepsilon}_a\sigma^- + \underline{\varepsilon}_a^*\sigma^+\right)\cdot i\,\mathcal{E}_r\left(\underline{\varepsilon}_r e^{-i\omega_r t} - \underline{\varepsilon}_r^* e^{+i\omega_r t}\right)$$