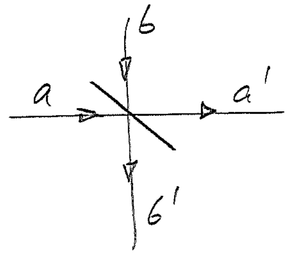
Last lecture:

- **Detector:** – Measures $\langle a^+ a \rangle$

  – Intensity or photon # counting.

- **Beam splitter:**



$$a' = U^+ a U = \cos\varphi/_2\, a + \sin\varphi/_2\, b$$

$$b' = U^+ b U = \cos\varphi/_2\, b - \sin\varphi/_2\, a$$

What is the interpretation of a in a propagating beam / detector?

Previously: Standing/propagating wave in box: $E(r,t)$.

Quantization: $\hat{E} = \mathcal{E}(r,t)(i a e^{ikr} + h.c.)$

with $\mathcal{E}(r,t) \equiv$ electric field per photon

Here: several possibilities:

– consider wave-packet: - $\mathcal{E}(r,t)$ field per photon in packet

– quantize energy going through area S in time T →

→ replace $V = L^3$ by $S \cdot c \cdot T$.

⇒ also gives signal in detector per unit time T!

⇒ signal measured in continuous beam is proportional to time T!

Action of a beam splitter:

Action on single input beam $|\psi\rangle|0\rangle$:

i) One photon:

$$\mathcal{U}|1,0\rangle = \mathcal{U}a^\dagger|0,0\rangle = \mathcal{U}a^\dagger\mathcal{U}^\dagger\underbrace{\mathcal{U}|0,0\rangle}_{=|0,0\rangle} = \mathcal{U}a^\dagger\mathcal{U}^\dagger|0,0\rangle$$

mode a, mode b

$$= \left(\cos\varphi/_2\, a^\dagger - \sin\varphi/_2\, b^\dagger\right)|0,0\rangle$$

$$= \cos\varphi/_2\,|1,0\rangle - \sin\varphi/_2\,|0,1\rangle$$

$\Rightarrow$ output is in a superposition of the photon in mode a' and b'

$\Rightarrow$ entangled state

Note: The avg. photon # in mode a' (b') is $\cos^2\varphi/_2$ ($\sin^2\varphi/_2$), equal to what is expected ~~for~~ a class. beam!

ii) Coherent state:

$$\mathcal{U}|\alpha,0\rangle = \mathcal{U}D_a(\alpha)\mathcal{U}^\dagger|0,0\rangle, \quad D_a(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$$

We have $\mathcal{U}D_a(\alpha)\mathcal{U}^\dagger \overset{\mathcal{U}\exp(A)\mathcal{U}^\dagger = \exp(\mathcal{U}A\mathcal{U}^\dagger)}{=} \exp(\alpha\,\mathcal{U}a^\dagger\mathcal{U}^\dagger - \alpha^*\mathcal{U}a\mathcal{U}^\dagger)$

$$= \exp\left(\alpha\left(\cos\varphi/_2\, a^\dagger - \sin\varphi/_2\, b^\dagger\right) - \alpha^*\left(\cos\varphi/_2\, a - \sin\varphi/_2\, b\right)\right)$$

$$= D_a(\alpha\cos\varphi/_2)\, D_b(-\alpha\sin\varphi/_2).$$

$$\Rightarrow \mathcal{U}|\alpha,0\rangle = D_a(\alpha\cos\varphi/2)\, D_b(-\alpha\sin\varphi/2)\,|0,0\rangle$$

$$= |\alpha\cos\varphi/2,\ -\alpha\sin\varphi/2\rangle.$$

$\Rightarrow$ coherent input is split into two __unentangled__ coherent output states $|\beta\rangle := |\alpha\cos\varphi/2\rangle$ and $|\gamma\rangle := |-\alpha\sin\varphi/2\rangle$, with $|\alpha|^2 = |\beta|^2 + |\gamma|^2$ (energy conservation!)

__Mixing of two inputs on a beam splitter:__

__Two photons: $|1,1\rangle$__:

$$\mathcal{U}|1,1\rangle = \mathcal{U}a^\dagger b^\dagger|0,0\rangle = \underbrace{\mathcal{U}a^\dagger\mathcal{U}^\dagger}\,\underbrace{\mathcal{U}b^\dagger\mathcal{U}^\dagger}\,\underbrace{\mathcal{U}|0,0\rangle}$$

$$= (\cos\varphi/2\, a^\dagger - \sin\varphi/2\, b^\dagger)(\cos\varphi/2\, b^\dagger + \sin\varphi/2\, a^\dagger)\,|0,0\rangle$$

$$= \Big[\underbrace{\sin\varphi/2\cos\varphi/2}_{(\sin\varphi)/2}(a^\dagger)^2 - \sin\varphi/2\cos\varphi/2\,(b^\dagger)^2 + \underbrace{(\cos^2\varphi/2 - \sin^2\varphi/2)}_{\cos\varphi}a^\dagger b^\dagger\Big]|0,0\rangle$$

$$= \sin\varphi\,\frac{|2,0\rangle - |0,2\rangle}{\sqrt{2}} + \cos\varphi\,|1,1\rangle$$

For a __balanced beam splitter__, $\varphi = \pi/2$:

$$\mathcal{U}|1,1\rangle = \frac{1}{\sqrt{2}}\left(|2,0\rangle - |0,2\rangle\right)$$

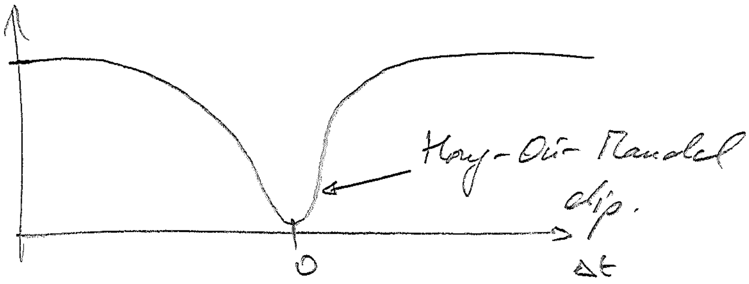
$\Rightarrow$ both photons always arrive at the same output

"Hong-Ou-Mandel effect" — photon "bunching"

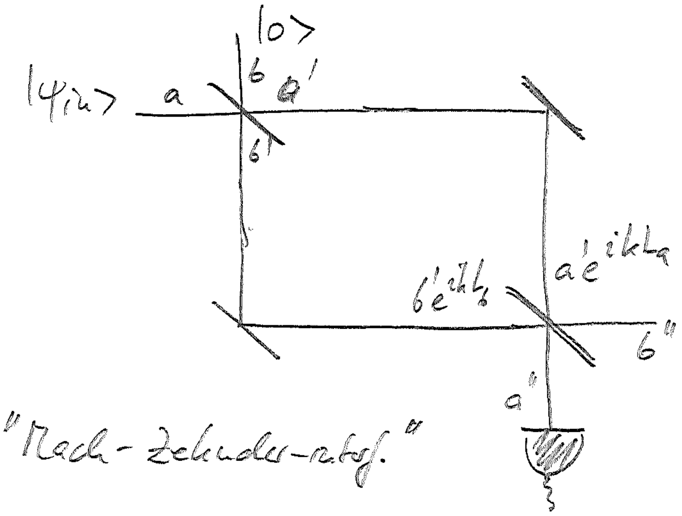
* **Reason:** Photons indistinguishable, and the two paths leading to $|1,1\rangle$ interfere destructively.

Note: Photons need to be indistinguishable $\Rightarrow$ wavepackets must match in space (modes) & time $\Rightarrow$ experimentally challenging.

Coincidence of photons in both detectors as a function of the delay Δt betw. them:



Interferometers:



"Mach-Zehnder-interf."

Restrict to 50/50 - BS

$$a \mapsto a' = \frac{1}{\sqrt{2}}(a+b)$$

$$b \mapsto b' = \frac{1}{\sqrt{2}}(a-b)$$

Paths differ by length $\Delta L = L_a - L_b$

$\Rightarrow$ phase difference : $\phi = k\Delta L$

$$a'' = \frac{1}{\sqrt{2}}\left(a'e^{ikL_a} + b'e^{ikL_b}\right) = \frac{1}{\sqrt{2}} \underbrace{\cancel{e^{ik\frac{L_a+L_b}{2}}}}_{\text{wlog}:\,=1} \left(a'e^{i\phi/2} + b'e^{-i\phi/2}\right)$$

$$= \frac{1}{2}\left((a+b)e^{i\phi/2} + (a-b)e^{-i\phi/2}\right)$$

$$= \cos\phi/2 \; a + i \sin\phi/2 \; b$$

Signal at output:

$$\langle \psi_{out} | (a'')^{+} a'' | \psi_{out} \rangle = \langle \psi_{in}, 0 | U^{+} (a'')^{+} a'' U | \psi_{in}, 0 \rangle$$

$$= \langle \psi_{in}, 0 | \left(U^{+} a''^{+} U \right) \left(U^{+} a'' U \right) | \psi_{in}, 0 \rangle$$

$$= \langle \psi_{in}, 0 | \left(\cos\phi/2 \, a^{+} - i \sin\phi/2 \, b^{+} \right) \left(\cos\phi/2 \, a + i \sin\phi/2 \, b \right) | \psi_{in}, 0 \rangle$$

$$= \langle \psi_{in}, 0 | \cos^2\phi/2 \, a^{+}a + \underbrace{\cos\phi/2 \sin\phi/2 \left(i a^{+} b - i b^{+} a \right)}_{\text{vanishes in } \langle 0 | \ldots | 0 \rangle} + \sin^2\phi/2 \, b^{+} b \, | \psi_{in}, 0 \rangle$$

$$= \frac{1}{2} (1 + \cos\phi) \underbrace{\langle \psi_{in} | a^{+} a | \psi_{in} \rangle}_{= \overline{n_{in}} \text{ : avg. \# photons in } |\psi_{in}\rangle} =: \overline{n_{out}}$$

⇒ Signal in interferometer independent of input state (as long as it does not fluctuate over $\Delta t = \frac{\Delta L}{c}$)

(In part.: States w/out fixed phase, e.g. Fock states, can be used!)

Photon # is discrete quantity: observed photon # fluctuates around $\overline{n_{out}}$ with variance

$$(\Delta n_{out})^2 = \langle \psi_{out} | a''^{+} a'' a''^{+} a'' | \psi_{out} \rangle - \overline{n_{out}}^2$$

with $c \equiv \cos\phi/2$, $s \equiv \sin\phi/2$:

$$= \langle \psi_{in}, 0 | (c a^{+} - i s b^{+})(c a + i s b)(c a^{+} - i s b^{+})(c a + i s b) | \psi_{in}, 0 \rangle - \overline{n_{out}}^2$$

$$= \langle \psi_{in}, 0 | c^4 a^{+} a a^{+} a + c^2 s^2 a^{+} a b b^{+} | \psi_{in}, 0 \rangle - \overline{n_{out}}^2$$

only non-vanishing terms in $\langle 0 | \ldots | 0 \rangle$

$$\ldots = c^4 \underbrace{\left(\langle \psi_{in} | a^\dagger a a^\dagger a | \psi_{in}\rangle - \left(\langle \psi_{in} | a^\dagger a | \psi_{in}\rangle\right)^2\right)}_{\text{variance of } \overline{n_{in}} :\ (\Delta n_{in})^2} + c^2 s^2 \underbrace{\langle \psi_{in} | a^\dagger a | \psi_{in}\rangle}_{\text{additional noise}}$$

variance of $\overline{n_{in}}$: $(\Delta n_{in})^2$
$\Rightarrow$ source is quantization of light into photons

additional noise from vac. fluctuations in input b!

Fock state $|n_n\rangle$:

$$(\Delta n_{out})^2 = s^2 c^2 n_{in} \implies \underline{\underline{\Delta n_{out} = \tfrac{1}{2} |\sin\phi| \sqrt{n_{in}}}}$$

$\Delta n_{in} = 0$

Coherent state $|\alpha\rangle$:

$$\langle \alpha | a^\dagger a a^\dagger a | \alpha\rangle = \langle \alpha | a^\dagger (a^\dagger a + 1) a | \alpha\rangle = |\alpha|^2 (|\alpha|^2 + 1)$$

$$\Rightarrow (\Delta n_{in})^2 = |\alpha|^2$$

$$(\Delta n_{out})^2 = c^4 |\alpha|^2 + c^2 s^2 |\alpha|^2 \implies \Delta n_{out} = |\cos \phi/2| \sqrt{\overline{n_{in}}}$$

Sensitivity of interferometer: Smallest $\Delta\phi$ s.th. change in signal $\overline{n_{out}}$ is larger than noise Δn_{out}:

$$\left| \frac{d\overline{n_{out}}}{d\phi} \right| \Delta\phi = \Delta n_{out}$$

We have $\dfrac{d\overline{n_{out}}}{d\phi} = -\dfrac{1}{2} \sin\phi\, \overline{n_{in}} \implies \Delta\phi = \dfrac{2 \Delta n_{out}}{|\sin\phi|\, \overline{n_{in}}}$

Fock state:

$$\Delta\phi = \frac{|\sin\phi|\sqrt{n_{in}}}{|\sin\phi|\,\overline{n}_{in}} = \frac{1}{\sqrt{n_{in}}} \quad (\text{indep. of } \phi!)$$

Coherent state:

$$\Delta\phi = \frac{2|\cos\phi/2|\sqrt{n_{in}}}{|\sin\phi|\,\overline{n}_{in}} \Rightarrow \text{minimal } \Delta\phi = \frac{1}{\sqrt{n_{in}}} \quad \left(\text{for } \phi = \pi, \text{ i.e. around dark setting}\right)$$

$\Rightarrow$ Same sensitivity as for Fock state!

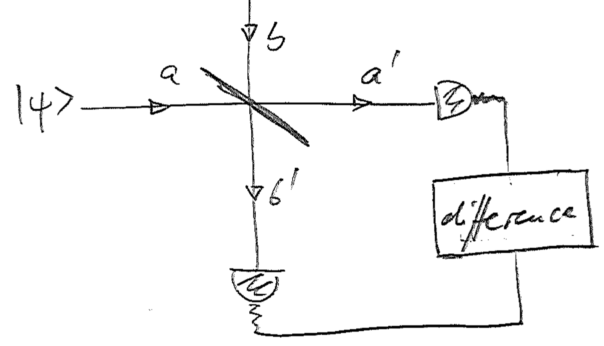
Accuracy can be increased by:

- increasing intensity of $|\alpha\rangle$ (or meas. time, if possible)
- reducing fluctuations in $|\psi_{in}\rangle$ and b - mode
  - $\Rightarrow$ squeezed vacuum in b can reduce fluctuations! ($\rightarrow$ Homework)

Homodyne detection:

Use information from both output ports:

$|\alpha\rangle$, $\alpha \equiv |\alpha| e^{i\phi}$



signal $= \langle a'^{\dagger}a' - b'^{\dagger}b' \rangle =: \Delta$

50/50:
$$a' = \frac{1}{\sqrt{2}}(a+b)$$
$$b' = \frac{1}{\sqrt{2}}(a-b)$$

$$a'^\dagger a' - b'^\dagger b' = \frac{1}{2}\left[(a^\dagger + b^\dagger)(a+b) - (a^\dagger - b^\dagger)(a-b)\right]$$

$$= a^\dagger b + b^\dagger a.$$

$$\underline{\underline{\Delta}} = \langle \psi, \alpha | a^\dagger b + b^\dagger a | \psi, \alpha \rangle = \langle \psi | \alpha a^\dagger + \alpha^* a | \psi \rangle$$

$$= |\alpha| \langle \psi | e^{-i\phi} a + e^{i\phi} a^\dagger | \psi \rangle = \underline{\underline{\sqrt{2}\, |\alpha| \langle \psi | X_\phi | \psi \rangle}}$$

⇒ Difference of signals measures quadrature X_ϕ of $|\psi\rangle$!
ϕ can be chosen by changing rel. phase of $|\psi\rangle$ & $|\alpha\rangle$.

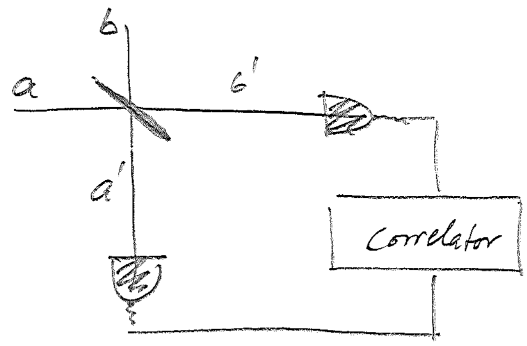
⟹ "<u>Homodyne detection</u>"

(Note: <u>Heterodyne detection</u>: freq. of $|\alpha\rangle \neq$ freq. of $|\psi\rangle$: output signal is $\langle \psi | X_{\phi + \Delta\omega t} | \psi \rangle$.)

Can we also measure e.g. $\langle X_\phi^2 \rangle$?

<u>Idea:</u> $\langle X_\phi^2 \rangle \hat{=} \langle (a'^\dagger a' - b'^\dagger b')^2 \rangle$

This can be measured by <u>correlating outcomes</u>:



<u>Classical:</u> signal $\hat{=} \langle |E_{a'}|^2 |E_{b'}|^2 \rangle$

<u>Q.M.:</u> signal $\hat{=} \langle \psi | a'^\dagger b'^\dagger b' a' | \psi \rangle$

→ <u>normal ordered product</u>!

(reasoning similar to detector: we remove 2 photons from $|\psi\rangle$).