Last lecture:

Coherent states (= quasi-classical states)

Unitary displacement operator

$$D(\alpha) = e^{\alpha a^+ - \alpha^* a}$$
$$= e^{-|\alpha|^2/2}\, e^{\alpha a^+}\, e^{-\alpha^* a}$$

It holds that

$$D^+(\alpha)\, a\, D(\alpha) = a + \alpha$$

$$D^+(\alpha)\, Q\, D(\alpha) = Q + \sqrt{2}\, \mathrm{Re}\,\alpha$$

$$D^+(\alpha)\, P\, D(\alpha) = P + \sqrt{2}\, \mathrm{Im}\,\alpha$$

i.e., D displaces states in phase space.

Coherent state:

$$|\alpha\rangle := D(\alpha)|0\rangle \quad \text{(displaced vacuum)}$$

with $a|\alpha\rangle = \alpha|\alpha\rangle$

$\langle\alpha|a^+a|\alpha\rangle = |\alpha|^2$; avg. photon # $= |\alpha|^2$.

Expansion of coherent state in Fock basis:

Use i) $D(\alpha) = e^{-|\alpha|^2/2} \underbrace{e^{\alpha a^\dagger}}_{= \sum \frac{\alpha^n}{n!}(a^\dagger)^n} \underbrace{e^{-\alpha^* a}}_{= \sum \frac{(-\alpha^*)^n}{n!} a^n}$

ii) $e^{-\alpha^* a}|0\rangle = \sum_{n=0} \frac{(-\alpha^*)^n}{n!} a^n|0\rangle = |0\rangle$

iii) $e^{\alpha a^\dagger}|0\rangle = \sum_{n=0} \frac{\alpha^n}{n!} \underbrace{a^{\dagger n}|0\rangle}_{= \sqrt{n}\sqrt{n-1}\ldots\sqrt{1}|n\rangle} = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$

$$\Rightarrow \quad |\alpha\rangle = D(\alpha)|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle .$$

Photon number distributed according to Poisson distribution

$$P_\alpha(n) = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} : \begin{cases} \langle n\rangle = |\alpha|^2 \\ (\Delta n)^2 = |\alpha|^2 \end{cases}$$

Overlap of coherent states:

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \qquad \text{(Homework)}$$

Completeness:

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d^2\alpha = \mathbb{1} \qquad \text{(Homework)}$$

$$\left(\text{cf. } \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1}\right).$$

Uncertainty: equal to uncertainty of vacuum $|0\rangle$
(since it is displaced vacuum)

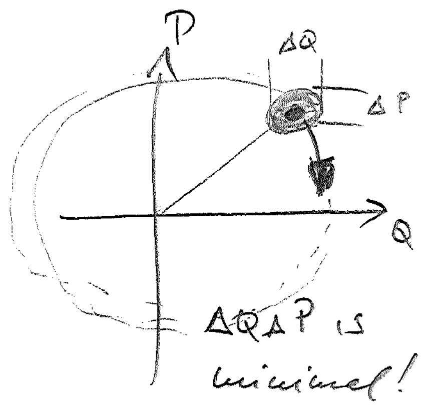
How do coherent states evolve under $H = \sum \hbar\omega(a^\dagger a + \frac{1}{2})$

$$|\alpha(t)\rangle = e^{-iHt/\hbar}|\alpha(0)\rangle = e^{-i\omega(a^\dagger a+\frac{1}{2})t} \sum_{n\geq 0} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$= e^{-i\omega/2\,t} \sum_{n\geq 0} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}|n\rangle = e^{-i\omega/2\,t}\,|\alpha(0)e^{-i\omega t}\rangle$$

$\Rightarrow \alpha(t) = \alpha(0)e^{-i\omega t} \Rightarrow$ this is exactly the evolution equation of the classical field!

$\Rightarrow$ Best approximation (min. uncertainty) to point in phase space.



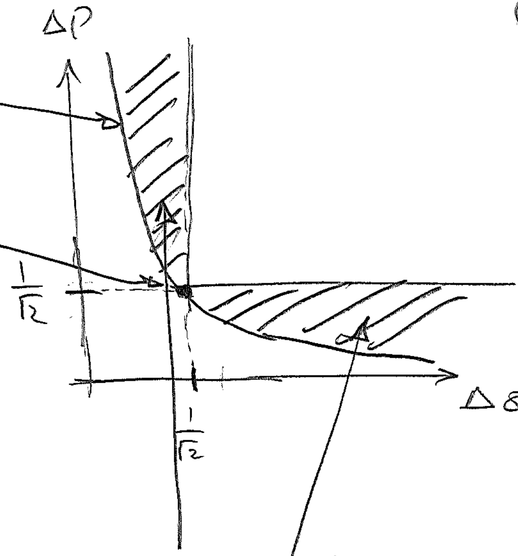
$\Delta Q \Delta P$ is minimal!

4. Squeezed states:

Uncertainty relation: $\Delta Q \Delta P \geq \frac{1}{2}$.

Coherent state: $\Delta Q = \Delta P = \frac{1}{\sqrt{2}}$

Squeezed states: states w/ reduced uncertainty in one direction, $\Delta X_\phi < 1/\sqrt{2}$.



states w/ red. uncertainty in one direction (below "standard quantum limit").

Squeezing operator:

$$S(\varepsilon) = \exp\left(\frac{\varepsilon^*}{2} a^2 - \frac{\varepsilon}{2} (a^+)^2\right); \quad \varepsilon = r e^{2i\phi}$$

We have $S^+(\varepsilon) = S(-\varepsilon) = S^{-1}(\varepsilon)$

$\Rightarrow S$ is unitary $\Rightarrow$ generated by Hamiltonian evolution,

$$H = -\frac{i}{2} \varepsilon (a^+)^2 + h.c. \quad \equiv \text{creation of photon pairs}$$

It holds $S^+(\varepsilon)\, a\, S(\varepsilon) = a \cosh r - a^+ e^{2i\phi} \sinh r$

(cf. exercise 1,4)

$S^+(\varepsilon)\, a^+ S(\varepsilon) = a^+ \cosh r - a\, e^{-2i\phi} \sinh r$

Homework!

It follows that $S^+(\varepsilon) X_\phi S(\varepsilon) = X_\phi e^{-r}$

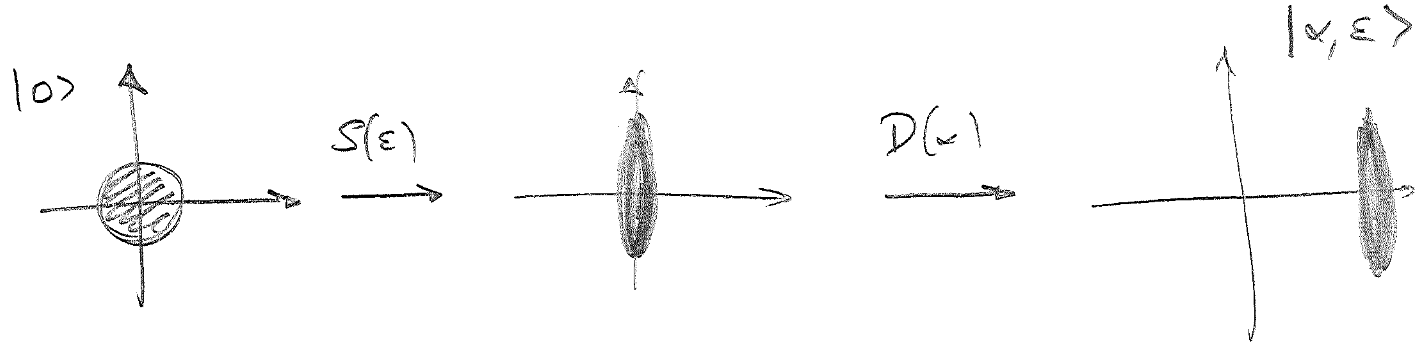
$$S^+(\varepsilon) X_{\phi+\pi/2} S(\varepsilon) = X_{\phi+\pi/2}\, e^{+r}$$

$S(\varepsilon)$ squeezes the state in direction X_ϕ and anti-squeezes it in the orthogonal direction ($\rightarrow \Delta X_\phi \Delta X_{\phi+\pi/2} \geq \frac{1}{2}$!)

$r = |\varepsilon|$ is called the squeeze factor.

Squeezed state:

$$|\alpha, \varepsilon\rangle = D(\alpha) \underbrace{S(\varepsilon)|0\rangle}_{\text{"squeezed vacuum"}}$$

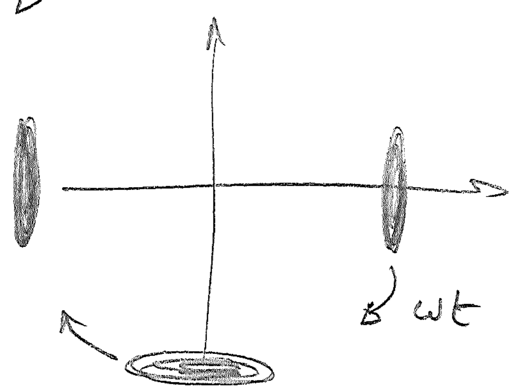


Squeezed state has "sub-standard" uncertainty in a given direction.

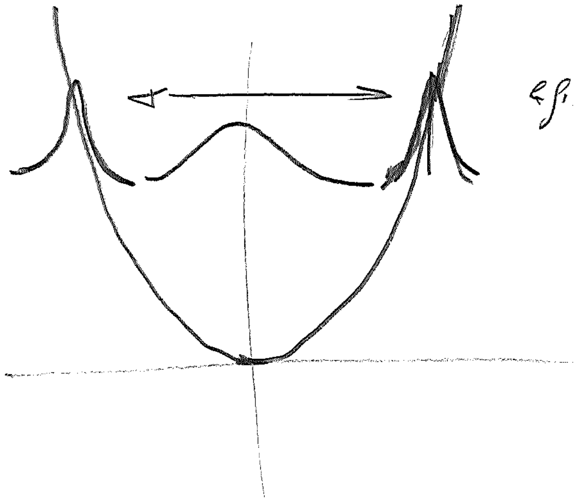
How does $|\alpha, \varepsilon\rangle$ evolve under $H = \hbar\omega(a^\dagger a + \frac{1}{2})$?

$\Rightarrow$ Heisenberg picture: rotate phase space coordinates X_ϕ

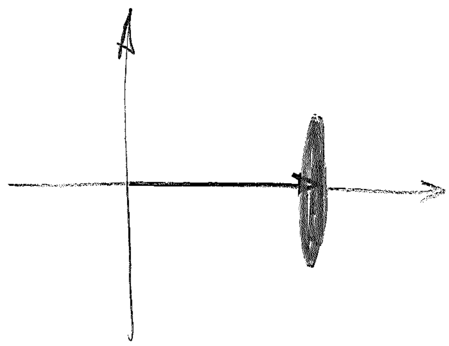
$\Rightarrow$



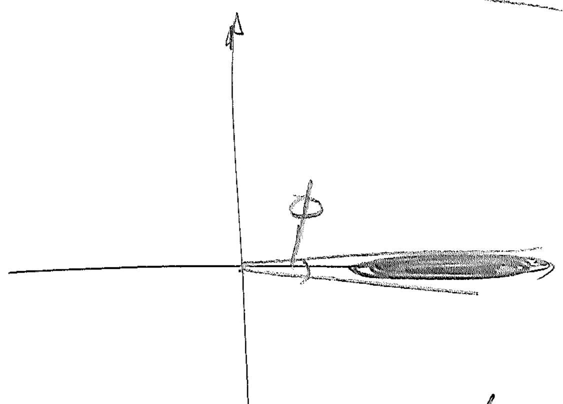
Classical picture for H.O.:



e.g. wavefunction more localized at ends / more spread in the middle!



"amplitude squeezed state"



"phase squeezed state"

Phase $\longleftrightarrow$ important in interferometric measurements.

phase sq. state important in high precision measurements to <u>decrease uncertainty</u> coming from quant. fluctuations!

III. Measurement of the light field

1. Photodetectors

* General idea: Incident photon is absorbed and ionizes atom. Signal is amplified & resulting current measured.

* Classical description:

  Resulting signal is proportional to field intensity:

$$S(t) \propto |E(r,t)|^2$$

field at the detector, assumed to vary slowly in space; we only measure slowly time-varying part – oscillations $\propto e^{-i\omega t}$ are averaged out.

* Quantized field:

  What does $|E(r,t)|^2$ correspond to?

$$\hat{E} = \underbrace{i\mathcal{E}(r)\, a\, e^{-i\omega t}}_{=\hat{E}^{(+)}} \underbrace{- i\mathcal{E}(r)\, a^{+} e^{i\omega t}}_{=\hat{E}^{(-)} \equiv \left(\hat{E}^{(+)}\right)^{+}}$$

$$= \hat{E}^{(+)} + \hat{E}^{(-)}$$

$$|\hat{E}(r,t)|^2 \longleftrightarrow \underbrace{\hat{E}^{(+)}\hat{E}^{(+)}}_{\substack{\propto e^{-2i\omega t} \\ \propto a \cdot a}} + \underbrace{\hat{E}^{(+)}\hat{E}^{(-)}}_{\propto a a^{+}} + \underbrace{\hat{E}^{(-)}\hat{E}^{(+)}}_{\propto a^{+} a} + \underbrace{\hat{E}^{(-)}\hat{E}^{(-)}}_{\substack{\propto e^{2i\omega t} \\ \propto a^{+} a^{+}}}$$

averaged out

* Detector annihilates photon:

State after detection is proportional to $a|\psi\rangle \propto E^{(+)}|\psi\rangle$,

and probability for this event is proportional to

$$\langle\psi| E^{(-)} E^{(+)} |\psi\rangle \equiv \langle E^{(-)} E^{(+)}\rangle \propto \langle a^\dagger a\rangle.$$

* Signal of photon detector is proportional to

$$\boxed{S(t) \propto \langle E^{(-)}(t)\, E^{(+)}(t)\rangle \propto \langle a^\dagger a\rangle}$$

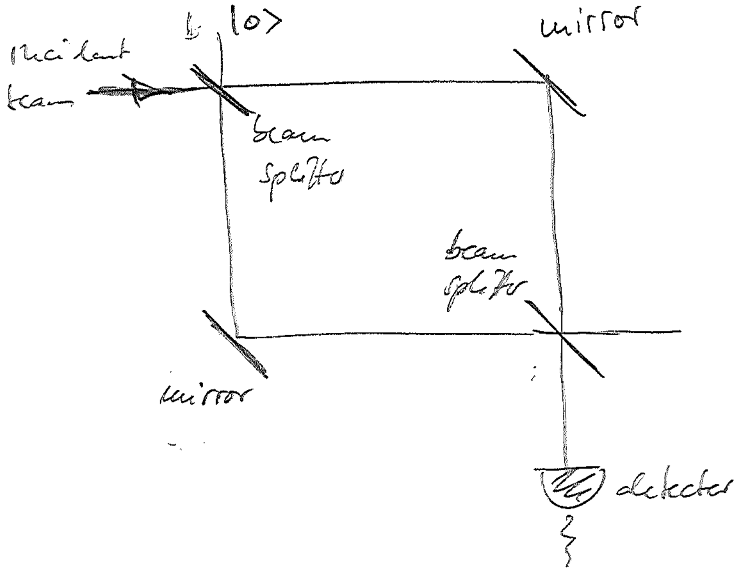
* This implies that the vacuum gives no signal.

* Dependent on design of photodetector – one could have detectors which measure $\langle a a^\dagger\rangle$ (though they are impractical).

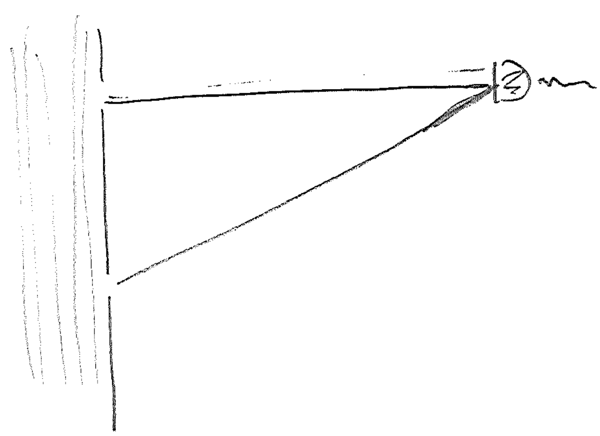
* Signal $\propto$ avg. photon number $\Rightarrow$ need interferometers to get more info about the state!

2. Interference experiments:

Interferometer:



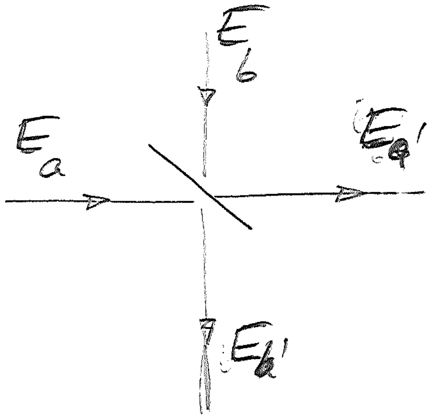
Double slit:



Beam splitters:

What is the action of a beam splitter?

Classical



$$E_{a'} = \cos\varphi \, E_a + \sin\varphi \, E_b$$
$$E_{b'} = \cos\varphi \, E_b - \sin\varphi \, E_a \quad , \text{ or}$$

$$\begin{pmatrix} E_{a'} \\ E_{b'} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi/2 & \sin\varphi/2 \\ -\sin\varphi/2 & \cos\varphi/2 \end{pmatrix}}_{V(\varphi)} \begin{pmatrix} E_a \\ E_b \end{pmatrix}$$

$\Longrightarrow V(\varphi)^\dagger V(\varphi) = \mathbb{1}$, i.e., V unitary $\Longleftrightarrow$

$\Longleftrightarrow$ ensures energy conservation: $|E_{a'}|^2 + |E_{b'}|^2 = |E_a|^2 + |E_b|^2$.

* QM beam splitter

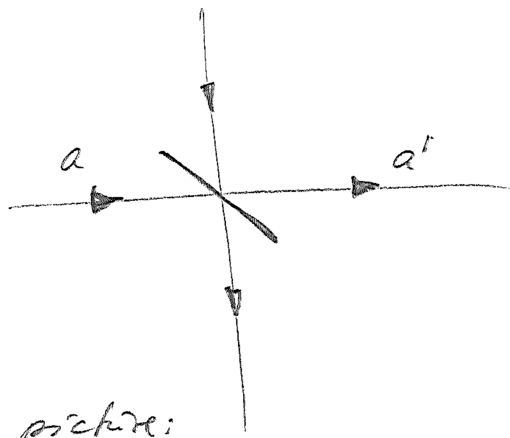
$$\hat{E}_{a'} = \cos\varphi/2\,\hat{E}_a + \sin\varphi/2\,\hat{E}_b\,; \qquad \hat{E}_a = \underbrace{i\mathcal{E} a e^{-i\omega t}}_{\hat{E}_a^{(+)}} + \text{h.c.}$$

$$\hat{E}_{b'} = \cos\varphi/2\,\hat{E}_b - \sin\varphi/2\,\hat{E}_a\,. \qquad \hat{E}_b = \underbrace{i\mathcal{E} b e^{-i\omega t}}_{\hat{E}_b^{(+)}} + \text{h.c.}$$

$\Rightarrow$ beam splitter transforms a, b as

$$a \longmapsto a' = \cos\varphi/2\; a + \sin\varphi/2\; b$$

$$b \longmapsto b' = \cos\varphi/2\; b - \sin\varphi/2\; a$$



Note: This transformation can be expressed as a unitary evolution in the Heisenberg picture:

$$a' = U^\dagger a U \qquad\qquad \left(\text{and} \quad (a')^\dagger = U^\dagger a^\dagger U\right.$$

$$b' = U^\dagger a U \qquad\qquad \left.\qquad\quad (b')^\dagger = U^\dagger b^\dagger U\right)$$

with $\quad U = \exp(-i/\hbar\, H_{ab} \cdot \varphi/2)$

where $\quad H_{ab} = \hbar\,(i a^\dagger b - i b^\dagger a)$

H_{ab} describes the Hamiltonian for a beam splitter, where a photon is transferred from mode a to mode b or vice versa.

In the Schrödinger picture, the beam splitter transforms the two-mode state $|\psi\rangle$ to $U|\psi\rangle$.