Reminder last lecture:

$$\underline{E}(\underline{r}) = \sum_{\underline{k},\lambda} \underline{\varepsilon}_{\underline{k},\lambda} \mathcal{E}_{\underline{k}} \left[i\alpha_{\underline{k}\lambda} e^{i\underline{k}\underline{r}} - i\alpha^*_{\underline{k}\lambda} e^{-i\underline{k}\underline{r}} \right]$$

with $\alpha(t) = e^{-i\omega_{\underline{k}} t}$ $\qquad (\omega_{\underline{k}} = c\underline{k})$

Quantization of electromagnetic field: replace $\alpha \to a$, $\alpha^* \to a^+$:

$$\hat{E}(\underline{r}) = \sum_{\underline{k},\lambda} \varepsilon_{\underline{k},\lambda} \mathcal{E}_{\underline{k}} \left[i a_{\underline{k}\lambda} e^{i\underline{k}\underline{r}} - i a^+_{\underline{k}\lambda} e^{-ikr} \right]$$

with $\left[a_{\underline{k}\lambda}, a^+_{\underline{k}'\lambda'}\right] = \delta_{\underline{k}\underline{k}'} \delta_{\lambda\lambda'}$

$\left[a_{\underline{k}\lambda}, a_{\underline{k}'\lambda'}\right] = 0$

$\left[a^+_{\underline{k}\lambda}, a^+_{\underline{k}'\lambda'}\right] = 0$

$$\mathcal{E}_{\underline{k}} = \sqrt{\frac{\hbar\omega_{\underline{k}}}{3\varepsilon_0 V}}$$

Hamiltonian: $\quad H = \sum_{\underline{k}\lambda} \hbar\omega_{\underline{k}} \left(a^+_{\underline{k}\lambda} a_{\underline{k}\lambda} + \frac{1}{2} \right)$

Interpretation: $\mathcal{E}_{\underline{k}}$: electric field per photon

$\hat{n}_{\underline{k},\lambda} = a^+_{\underline{k},\lambda} a_{\underline{k},\lambda}$: photon # operator for mode $(\underline{k},\lambda)$

$a^+_{\underline{k},\lambda}$, $a_{\underline{k},\lambda}$: create/annihilate photon in mode $(\underline{k},\lambda)$

Electric/magnetic field replaced by field operators $\hat{E}, \hat{B}$:

System in state $|\psi\rangle \Rightarrow$ exp. value of field is

$$\langle\psi|\hat{E}(r)|\psi\rangle \ , \ \langle\psi|\hat{B}(r)|\psi\rangle .$$

Schrödinger picture:

- $a_{k,\lambda}$ time independent
- state $|\psi\rangle$ evolves under H: $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$,

$$\text{or } \frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}H(t)|\psi(t)\rangle$$

Heisenberg picture:

- state $|\psi\rangle$ time independent
- operators evolve in time:

$$a_{k,\lambda}(t) = e^{iHt/\hbar}\,a_{k,\lambda}(0)\,e^{-iHt/\hbar} \overset{\text{HW}}{=} e^{-i\omega t}\,a_{k,\lambda}(0)$$

$\longrightarrow$ resembles classical field!

$$\left(\text{or: } \frac{\partial}{\partial t}a_{k,\lambda}(t) = -\frac{i}{\hbar}\left[a_{k,\lambda}(t), H\right]\right)$$

Remarks:

* Choice of normal modes for quantization:
  - Many normal modes for quantization:
    * propagating waves
    * standing waves
    * circularly polarized waves

      ⋮
  - Best choice dep. on situation ($\longleftrightarrow$ class. solutions)
  - Properties of "photons" (e.g. well def. momentum / angular mom., ...) dep. on basis

* Continuum: $\sum_k \longrightarrow \int_k d^3k$ (cf. later)

* "Particle-like" ($\equiv$ localized) photons:

  $\longrightarrow$ wavepacket $\int dw(k)\, e^{ikx} =$ 

  $\longrightarrow$ $a^+ = \int dk\, w(k)\, a_k^+$ creates localized photon

II. States of the electromagnetic field

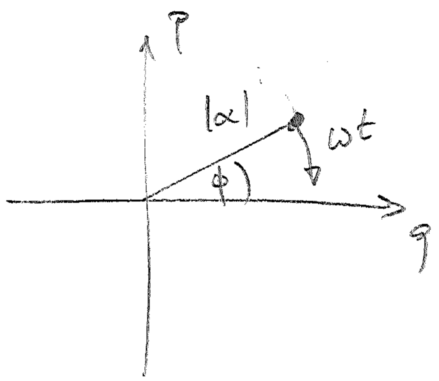
We focus for now on a single mode $(\underline{k},\lambda)$: omit subscripts!

classical: $\underline{E}(\underline{r},t) = \underline{\mathcal{E}}\left[i\alpha(t)e^{i\underline{k}\underline{r}} + c.c.\right]$

$$\hat{\underline{E}}(r) = \underline{\mathcal{E}}\left[i\,a\,e^{i\underline{k}\underline{r}} + h.c.\right]$$

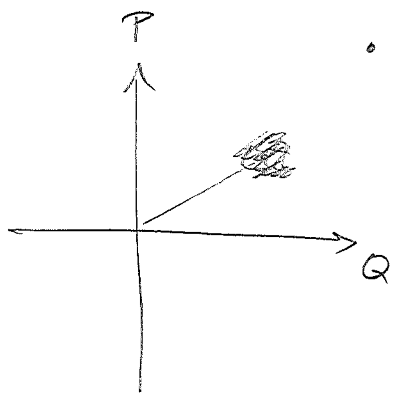
1. Phase space:

State of class. H.O. characterized by canonical coordinates $q = \mathrm{Re}\,\alpha$; $p = \mathrm{Im}\,\alpha$ $\Rightarrow$ phase space representation:



- classical system = point in phase space.
- $|\alpha| \hat{=}$ amplitude of EM field
- ϕ: phase/oscillation

QM H.O.: $Q = \frac{a+a^\dagger}{\sqrt{2}} \hat{=} \mathrm{Re}\,\alpha$; $P = \frac{i(a^\dagger - a)}{\sqrt{2}} \hat{=} \mathrm{Im}\,\alpha$; $[Q,P] = i$



- Uncertainty relation $(\Delta P)\cdot(\Delta Q) \geq \frac{1}{2}$

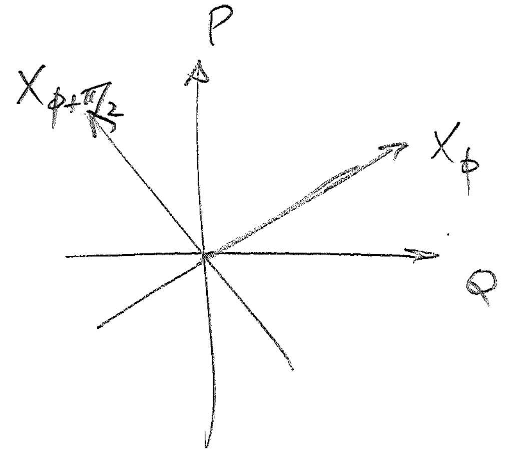
$\Rightarrow$ quantum state $\hat{=}$ distribution in phase space. (will be made more precise later).

- Time evolution:
  - Schrödinger picture (on state): "cloud" rotates w/ $-\omega t$.

* Heisenberg picture (on operators):
coordinate system rotates w/ $+\omega t$

→ General phase space coordinates

$$X_\phi = \frac{e^{-i\phi} a + e^{i\phi} a^\dagger}{\sqrt{2}}$$



$$Q = X_0;\ P = X_{\pi/2};\ [X_\phi, X_{\phi+\pi/2}] = i.$$

General uncertainty relation:

$$\Delta X_\phi \, \Delta X_{\phi+\pi/2} \geq \frac{1}{2} \left| \langle [X_\phi, X_{\phi+\pi/2}] \rangle_\psi \right| = \frac{1}{2}$$

(With variance $\Delta O = \sqrt{\langle (O - \langle O \rangle)^2 \rangle} = \sqrt{\langle O^2 \rangle - \langle O \rangle^2}$, where $\langle \cdot \rangle = \langle \cdot \rangle_\psi = \langle \psi | \cdot | \psi \rangle$)

(Preliminary) interpretation of phase space distribution:
Projection onto axis ≡ prob. distr. of ψ in that basis.
E.g.: Proj. onto Q-Axis ≡ $|\psi(r)|^2$

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2. Fock states:

Fock state: $|n\rangle \equiv n$ photons; $H = \hbar\omega(\hat{n} + \frac{1}{2})$

Energy: $\langle n | H | n \rangle = \hbar\omega \langle n | \hat{n} + \frac{1}{2} | n \rangle = \hbar\omega (n + \frac{1}{2})$

Average field: $\langle n | X_\phi | n \rangle = \frac{1}{\sqrt{2}} \langle n | e^{-i\phi} a + e^{i\phi} a^\dagger | n \rangle$

$$= \frac{1}{\sqrt{2}} e^{-i\phi} \underbrace{\langle n | a | n \rangle}_{\substack{\propto |n-1\rangle \\ \equiv 0}} + \frac{1}{\sqrt{2}} e^{i\phi} \underbrace{\langle n | a^\dagger | n \rangle}_{\substack{\propto |n+1\rangle \\ \equiv 0}} = 0.$$

→ Avg. field is zero.

Note: This is the electric field w/out the time dependence $e^{-i\omega t}$!

Variance: $\langle n|(X_\phi - \underbrace{\langle n|X_\phi|n\rangle}_{=0})^2|n\rangle = \langle n|X_\phi^2|n\rangle$

$$= \frac{1}{2}\langle n| e^{-2i\phi}a^2 + \underbrace{a a^\dagger}_{=n+1} + \underbrace{a^\dagger a}_{=n} + e^{2i\phi}(a^\dagger)^2 |n\rangle$$

$$= \frac{1}{2}(2n+1) = \underline{\underline{n+\tfrac{1}{2}}}.$$

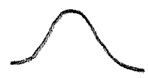
Variance $\underline{\underline{\langle n|X_\phi^2|n\rangle = n+\tfrac{1}{2}}}$; $(\Delta X_\phi) = \sqrt{n+\tfrac{1}{2}}$

Uncertainty: $\Delta Q \Delta P = n + \frac{1}{2} \geq \frac{1}{2}$.

- For $n \geq 1$, uncertainty is larger than required,
- The $n=0$ state is a <u>minimum uncertainty state</u>!

Phase space distribution:

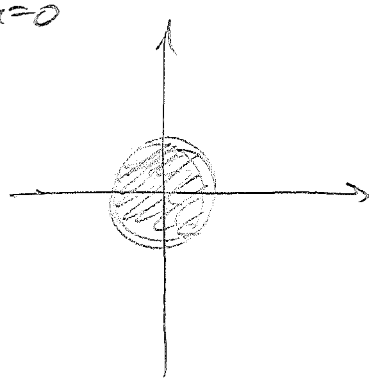
- Proj. onto x axis is solution of H.O.:

  $n=0$: 

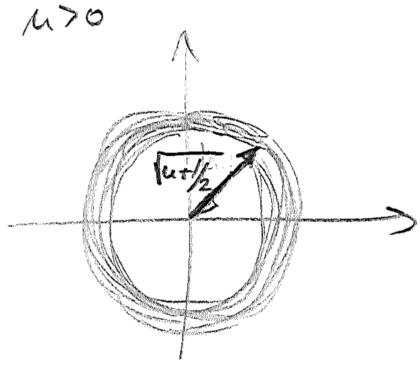
  larger n: 

- Eigenstate → time indep. → rotationally invariant (Heis. picture).

$n=0$



$n>0$



3. Coherent states (Glauber states):

→ will see: closest to "classical states"

Displacement operator:

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$$

Note: exponential of operator def. via Taylor series:

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Important identities (cf. Exercise 1.4) for $[A,[A,B]] = [B,[A,B]] = 0$:

$$e^A B e^{-A} = B + [A,B] \qquad (*)$$

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \qquad (**)$$

equiv.: $D(\alpha) \overset{(**)}{=} e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-|\alpha|^2/2}$ (as $[\alpha a^\dagger, -\alpha^* a] = |\alpha|^2$).

Properties of $D(\alpha)$:

$$D(0) = \mathbb{1}$$

$$D^\dagger(\alpha) = \exp(\alpha^* a - \alpha a^\dagger) = D(-\alpha) = D^{-1}(\alpha)$$

(⇒ $D(\alpha)$ is unitary ⇒ it can be created by a physical process!

$e^{-iHt/\hbar} = D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$ ⇒ $H = i\hbar(\alpha a^\dagger - \alpha^* a)$

⇒ absorption/emission of photons by cpl. to class. beam w/ strength α.)

$$\underbrace{D^\dagger(\alpha)}_{=D(-\alpha)} a D(\alpha) \overset{(*)}{=} a + [-\alpha a^\dagger + \alpha^* a, a] = a + \alpha$$

$$D^\dagger(\alpha) a^\dagger D(\alpha) = a^\dagger + \alpha^*$$

$$D^\dagger(\alpha)\, Q\, D(\alpha) = \frac{1}{\sqrt{2}} D^\dagger(\alpha)\left(a+a^\dagger\right) D(\alpha)$$

$$= \frac{1}{\sqrt{2}}\left(a+\alpha+a^\dagger+\alpha^*\right)$$

$$= Q + \sqrt{2}\,\mathrm{Re}\,\alpha$$

$$D^\dagger(\alpha)\, P\, D(\alpha) = P + \sqrt{2}\,\mathrm{Im}\,\alpha$$

$\Rightarrow D(\alpha)$ displaces states in phase space (by $\sqrt{2}\alpha$)

$$D(\alpha+\beta) = e^{-i\,\mathrm{Im}(\alpha\beta^*)} D(\alpha) D(\beta)$$

Coherent State = Displaced vacuum

$$|\alpha\rangle = D(\alpha)|0\rangle$$

Fock state $|0\rangle \equiv$ vacuum

Properties:

$$\underline{a|\alpha\rangle} = a D(\alpha)|0\rangle = D(\alpha) D^\dagger(\alpha)\, a\, D(\alpha)|0\rangle$$

$$= D(\alpha)(a+\alpha)|0\rangle = \alpha D(\alpha)|0\rangle = \underline{\underline{\alpha|\alpha\rangle}}$$

$\Rightarrow |\alpha\rangle$ eigenstate of a w/ eigenvalue $\alpha \in \mathbb{C}$.

(i.e.: We can remove a photon from $|\alpha\rangle$ w/out changing it

$\rightarrow$ "classical" feature.

Note: The equation $a|\alpha\rangle = \alpha|\alpha\rangle$ can be used to define $|\alpha\rangle$.)

<u>Also</u>: $\langle\alpha| a^\dagger = (a|\alpha\rangle)^\dagger = \langle\alpha|\alpha^*$

(<u>But</u>: $a^\dagger|\alpha\rangle \not\propto |\alpha\rangle$!)

$\Rightarrow \langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$

$\Rightarrow$ avg. photon number α!