I. Quantization of the light field

1. The quantum mechanical harmonic oscillator

Classical harmonic oscillator:

Hamiltonian $$H = \frac{1}{2m} p_0^2 + \frac{1}{2} m\omega^2 q_0^2$$

Conjugate variables p_0, q_0: $\dot{q}_0 = \frac{\partial H}{\partial p_0}$; $\dot{p}_0 = -\frac{\partial H}{\partial q_0}$

Quantization: Replace p_0, q_0 by operators $\hat{p}_0, \hat{q}_0$

with $[\hat{q}_0, \hat{p}_0] = i\hbar$ (e.g.: $\hat{q}_0 = x$; $\hat{p}_0 = -i\hbar\frac{\partial}{\partial x}$)

Quantum mech. H.O.:

$$H = \frac{1}{2m}\hat{p}_0^2 + \frac{1}{2} m\omega^2 \hat{q}_0^2$$

Natural units: $\hat{p} = \frac{1}{\sqrt{m\omega\hbar}}\hat{p}_0$; $\hat{q} = \sqrt{\frac{m\omega}{\hbar}}\,\hat{q}_0$

$$\Rightarrow H = \hbar\omega \frac{\hat{p}^2 + \hat{q}^2}{2} \; ; \; [\hat{q}, \hat{p}] = i$$

(Note: Different normalization for natural units exist, e.g. with $[\hat{q}, \hat{p}] = \frac{i}{2}$).

Omit $\hat{}$ from now on!

Solution ("algebraic method"):

Define $a = \frac{q+ip}{\sqrt{2}}$; $a^+ = \frac{q-ip}{\sqrt{2}}$.

Then: $q = \frac{a^+ + a}{\sqrt{2}}$; $p = \frac{i(a^+ - a)}{\sqrt{2}}$

$$\underline{[a, a^+]} = \frac{1}{2}[q+ip, q-ip] = \frac{1}{2}\Big(\underbrace{[q,q]}_{=0} + i\underbrace{[p,q]}_{=-i} - i\underbrace{[q,p]}_{=i} + \underbrace{[p,p]}_{=0}\Big)$$

$$\underline{= \mathbb{1}.} \qquad (\Leftrightarrow \; aa^+ = 1 + a^+a)$$

Since $a^+a = \frac{1}{2}(q-ip)(q+ip) = \frac{1}{2}\Big(q^2+p^2+i\underbrace{(qp-pq)}_{=[q,p]=i}\Big)$

$$= \frac{q^2+p^2}{2} - \frac{1}{2}$$

$$\Rightarrow \quad \underline{H = \hbar\omega\left(a^+a + \tfrac{1}{2}\right).}$$

Define $\hat{n} = a^+a$. Then $\hat{n}a = a^+aa = (aa^+-1)a = a(\hat{n}-1)$.

i.e.: Given eigenstate $\hat{n}|n\rangle = n|n\rangle$

$$\Rightarrow \quad \hat{n}(a|n\rangle) = a(\hat{n}-1)|n\rangle = (n-1)(a|n\rangle)$$

$$\Rightarrow \quad \frac{a|n\rangle}{\|a|n\rangle\|} =: |n-1\rangle \text{ is eigenstate w/ eigenvalue } n-1.$$

H is lower bounded (no states below min of potential)

$\Rightarrow \exists \; |n_0\rangle$ s.th. $a|n_0\rangle = 0$.

For this $|n_0\rangle$: $a^\dagger \underbrace{a|n_0\rangle}_{=0} = 0 = 0|n_0\rangle \Rightarrow \underline{n_0 = 0.}$

$\Rightarrow n = 0, 1, 2, \dots$

Fock states $|n\rangle$, $n = 0, 1, 2, \dots$

$\hat{n} = a^\dagger a$; $\hat{n}|n\rangle = n|n\rangle$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$ "lowering/annihilation operator"

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ "raising/creation operator"

$$H|n\rangle = \underbrace{\hbar\omega\left(n+\tfrac{1}{2}\right)}_{\equiv E_n}|n\rangle$$

Completeness relation: $\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1}$

Orthogonality: $\langle n|m\rangle = \delta_{nm}$

Particle interpretation:

$|0\rangle$: vacuum

$a^\dagger$: creates particle ("phonon")

a : annihilates particle

$|n\rangle$: state with n phonons

$\hat{n} = a^\dagger a$: particle number operator

Two or more uncoupled H.O.s:

$$H = \hbar\omega_1 \frac{p_1^2+q_1^2}{2} + \hbar\omega_2 \frac{p_2^2+q_2^2}{2} + \ldots + \hbar\omega_K \frac{p_K^2+q_K^2}{2}$$

quantize: p_j, q_j operators with $[p_j, q_k] = i\delta_{jk}$

Creation/annihilation ops:

$$a_k = \frac{q_k + ip_k}{\sqrt{2}} \; ; \quad a_k^+ = \frac{q_k - ip_k}{\sqrt{2}}$$

$$[a_k, a_l] = 0; \quad [a_k^+, a_l^+] = 0; \quad [a_k, a_l^+] = \delta_{kl}$$

bosonic commutation relations

Particle # ops:

$\hat{n}_k = a_k^+ a_k$: # phonons in H.O. ("mode") k

$$H = \sum_{k=1}^{K} \hbar\omega_k \left(\hat{n}_k + \frac{1}{2}\right)$$

Fock basis: $|n_1, n_2, \ldots, n_K\rangle = |n_1\rangle . |n_2\rangle \cdot \ldots \cdot |n_K\rangle$

$$\hat{n}_k |n_1, n_2, \ldots\rangle = n_k |n_1, n_2, \ldots\rangle$$

$$a_k |\ldots, n_{k-1}, n_k, n_{k+1}, \ldots\rangle = \sqrt{n_k}\, |\ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots\rangle$$

General state of K H.O.s:

$$|\psi\rangle = \sum c_{n_1, n_2, \dots, n_K} |n_1, n_2, \dots, n_K\rangle$$

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2. Quantization of the electromagnetic field

Maxwell Eqs. in vacuum:

$$\nabla \cdot \underline{E}(\underline{r},t) = 0 \qquad (i)$$

$$\nabla \cdot \underline{B}(\underline{r},t) = 0 \qquad (ii)$$

$$\nabla \times \underline{E}(\underline{r},t) = -\dot{\underline{B}}(\underline{r},t) \qquad (iii)$$

$$\nabla \times \underline{B}(\underline{r},t) = \frac{1}{c^2}\dot{\underline{E}}(\underline{r},t) \qquad (iv)$$

For simplicity, we consider finite volume $V = L \times L \times L$ with periodic boundaries.

Solutions can be decomposed in plane waves:

From Eqs. (i), (ii):

$$E(\underline{r},t) = \sum_{\underline{k},\lambda} \underline{\varepsilon}_\lambda \tilde{E}_{\underline{k}\lambda}(t)\, e^{i\underline{k}\cdot\underline{r}}$$

$$B(\underline{r},t) = \sum_{\underline{k},\lambda} \underline{\varepsilon}'_\lambda \tilde{B}_{\underline{k},\lambda}(t)\, e^{i\underline{k}\cdot\underline{r}}$$

$$\underline{k} = (k_x, k_y, k_z); \quad k_\alpha = \frac{2\pi n_\alpha}{L} \quad (\alpha = x, y, z)$$

$\underline{\varepsilon}_\lambda \perp \underline{k}$; $\lambda = 1,2$; $\underline{\varepsilon}_1 \perp \underline{\varepsilon}_2$;

$$\underline{\varepsilon}'_\lambda = \frac{\underline{k}}{|\underline{k}|} \times \underline{\varepsilon}_\lambda$$

Remark: Dep. on scenario, decompositions in another basis might be more appropriate.

Eqs. (iii), (iv), + fields must be real:

$$\underline{E}(\underline{r},t) = \sum_{\underline{k},\lambda} \underline{\varepsilon}_\lambda \, \mathcal{E}_{\underline{k}} \left[i\alpha_{\underline{k},\lambda}(t) e^{i\underline{k}\cdot\underline{r}} - i\alpha^*_{\underline{k},\lambda}(t) e^{-i\underline{k}\cdot\underline{r}} \right]$$

$$\underline{B}(\underline{r},t) = \sum_{\underline{k},\lambda} \underline{\varepsilon}'_\lambda \frac{\mathcal{E}_{\underline{k}}}{c} \underbrace{\left[i\alpha_{\underline{k},\lambda}(t) e^{i\underline{k}\cdot\underline{r}} - i\alpha^*_{\underline{k},\lambda}(t) e^{-i\underline{k}\cdot\underline{r}} \right]}_{(*)}$$

with $\alpha_{\underline{k},\lambda}(t) = \alpha_{\underline{k},\lambda}(0) e^{-i\omega_k t}$; $\omega_{\underline{k}} = c \cdot \underline{k}$

$$\left(\text{i.e.: } (*) = 2|\alpha_{\underline{k},\lambda}(0)| \cos\left(-\omega_k t + \underline{k}\cdot\underline{r} + \varphi_{k,\lambda} \right) \right)$$

We choose $\mathcal{E}_{\underline{k}} = \sqrt{\dfrac{\hbar \omega_{\underline{k}}}{2\varepsilon_0 V}}$

Hamiltonian:

$$H = \frac{\varepsilon_0}{2} \int_V d^3\underline{r} \left[\underline{E}^2(\underline{r},t) + c^2 \underline{B}^2(\underline{r},t) \right]$$

$$= \dots = 2\varepsilon_0 V \sum_{\underline{k},\lambda} \mathcal{E}_{\underline{k}}^2 \, |\alpha_{\underline{k},\lambda}|^2$$

$$= \sum_{\underline{k},\lambda} \hbar\omega_{\underline{k}} \underbrace{|\underline{\alpha}_{k,\lambda}|^2}_{=(\mathrm{Re}\,\alpha_{\underline{k},\lambda})^2 + (\mathrm{Im}\,\alpha_{\underline{k},\lambda})^2}$$

$q_{k,\lambda} = \sqrt{2\hbar}\,\mathrm{Re}\,\alpha_{\underline{k},\lambda}$ and $p_{\underline{k},\lambda} = \sqrt{2\hbar}\,\mathrm{Im}\,\alpha_{\underline{k},\lambda}$

form conjugate variables:

$$\underline{\underline{\frac{dH}{dp_{\underline{k},\lambda}}}} = \sqrt{2\hbar}\,\omega_{\underline{k}}\,\mathrm{Im}\,\alpha_{\underline{k},\lambda} = \sqrt{2\hbar}\,\frac{d}{dt}\mathrm{Re}\,\alpha_{k\lambda} = \underline{\underline{\frac{d}{dt} q_{\underline{k},\lambda}}}$$

since $\alpha_{\underline{k},\lambda} = e^{-i\omega_{\underline{k}} t}$

and similarly $\frac{dH}{dq_{\underline{k},\lambda}} = -\frac{d}{dt} p_{\underline{k},\lambda}$.

Quantization: Operators $\hat{q}_{k,\lambda}, \hat{p}_{k,\lambda}$ with $[\hat{q}_{k,\lambda}, \hat{p}_{k,\lambda}] = i\hbar$

$$\Rightarrow \quad a_{k,\lambda} = \frac{\hat{q}_{k,\lambda} + i\hat{p}_{k,\lambda}}{\sqrt{2\hbar}} \; ; \; a^+_{k,\lambda} = \frac{\hat{q}_{k,\lambda} - i\hat{p}_{k\lambda}}{\sqrt{2\hbar}}$$

$\Rightarrow$ Replace $\alpha_{\underline{k},\lambda} \longrightarrow a_{\underline{k},\lambda}$

$\alpha^*_{\underline{k},\lambda} \longrightarrow a^\dagger_{\underline{k},\lambda}$

<u>Quantized EM field:</u>

$$\hat{\underline{E}}(\underline{r}) = \sum_{\underline{k},\lambda} \underline{\varepsilon}_\lambda \, \mathcal{E}_k \left[i a_{\underline{k}\lambda} e^{i\underline{k}\cdot\underline{r}} - i a^\dagger_{\underline{k}\lambda} e^{-i\underline{k}\cdot\underline{r}} \right]$$

$$\hat{\underline{B}}(\underline{r}) = \sum_{\underline{k},\lambda} \underline{\varepsilon}'_\lambda \, \frac{\mathcal{E}_k}{c} \left[i a_{\underline{k}\lambda} e^{i\underline{k}\cdot\underline{r}} - i a^\dagger_{\underline{k}\lambda} e^{-i\underline{k}\cdot\underline{r}} \right]$$

with

$$[a_{\underline{k}\lambda}, a^\dagger_{\underline{k}'\lambda'}] = \delta_{\underline{k}\underline{k}'} \delta_{\lambda\lambda'}$$

$$[a_{\underline{k}\lambda}, a_{\underline{k}'\lambda'}] = 0; \quad [a^\dagger_{\underline{k}\lambda}, a^\dagger_{\underline{k}\lambda}] = 0.$$

$$\mathcal{E}_\ell = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$$

<u>Hamiltonian:</u> $H = \sum_{\underline{k}\lambda} \hbar\omega_{\underline{k}} \left(a^\dagger_{\underline{k}\lambda} a_{k\lambda} + \frac{1}{2} \right)$

<u>Interpretation:</u> $\mathcal{E}_{\underline{k}}$: electric field per photon

$\hat{n}_{\underline{k},\lambda} = a^\dagger_{k,\lambda} a_{k,\lambda}$: photon # operator for mode $\underline{k},\lambda$

$a^\dagger_{k,\lambda} / a_{k,\lambda}$: create / destroy photon in mode