Lecture "Quantum Optics" — Exercise Sheet #13

Problem 1 (lengthy)

As discussed in the lecture (pg. 108 of the lecture notes), N coupled harmonic oscillators with Hamiltonian

$$H = \frac{1}{2M}\sum_{n=1}^{N} p_n^2 + \frac{1}{2}M\nu^2 \sum_{n,m=1}^{N} A_{n,m} q_n q_m$$

can be brought into diagonal form by diagonalizing the real symmetric matrix $A_{n,m} = \sum \mu_k b_n^k b_m^k$ (note that b_n^k forms an orthonormal basis) by changing to coordinates

$$\tilde{q}_k = \sum_n b_n^k q_n \ ; \qquad \tilde{p}_k = \sum_n b_n^k p_n \ .$$

Verify that the resulting Hamiltonian describes decoupled harmonic oscillators (use the orthogonality relations for the b_n^k), and that the new $\tilde{q}_k$ and $\tilde{p}_k$ are still conjugate variables (both classically, by checking that they still verify Hamilton's equations, and quantum mechanically, by checking that they preserve the canonical commutation relations).

Determine $A_{n,m}$ for the case of two ions in a harmonic trap (where the trap potential is $\frac{1}{2}M\nu^2x^2$, and the ions interact via the Coulomb interaction), and determine the normal modes and their frequencies by diagonalizing $A_{n,m}$. (Note that the q_n are defined relative to the equilibrium position of the ions. Also, the Hamiltonian is of course only of the form above up to an additive constant. Note that you need to expand the Couloumb interaction to second order in $q_2 - q_1$.)

Problem 2 (medium)

The aim of this problem is to derive an alternative way to perform a step (ii) in the Cirac-Zoller gate, i.e., the controlled-Z between the vibrational CM mode and the one ion.

To this end, we need to consider an ion in a trap in one of the four states $|g;0\rangle$, $|g;1\rangle$, $|e;0\rangle$, or $|e;1\rangle$ (the first system is the internal state of the ion and the second the state of the vibrational mode). Applying a 2π-pulse on the $|g;1\rangle \leftrightarrow |e;0\rangle$ (red sideband) transition was not an option, as the state $|e;1\rangle$ would "leak out" into $|g;2\rangle$ due to the different Rabi frequency of that transition.

This can be overcome by using rotations about both the X and the Y axis, which can be achieved by tuning the phase of the applied laser. Consider the rotations $R_x(\phi) = \exp[-i\phi\sigma_x/2]$ and $R_y(\phi) = \exp[-i\phi\sigma_y/2]$. Verify that the pulse sequence

$$R_y(\phi)R_x(\pi)R_y(\phi)R_x(\pi) = -\mathbb{1} \ ,$$

for any ϕ, and equivalently for ϕ and π interchanged. The latter implies that for $\phi = \pi/\sqrt{2}$,

$$R_y(\phi\sqrt{2})R_x(\pi\sqrt{2})R_y(\phi\sqrt{2})R_x(\pi\sqrt{2}) = -\mathbb{1} \ .$$

Use this to show that this pulse sequence can be used to implement a non-trivial controlled-phase gate between within the four-dimensional subspace used by the system.

(Note: In this setup, when assembling the three steps the -1 phase is applied to the $|g_1;e_2\rangle$ state. You can easily convince yourself that this can be changed easily to $|e_1;e_2\rangle$ by applying a π pulse to the bare transition of the first ion at the beginning and end of the sequence!)

Problem 3 (easy)

Show that we can fully determine the state ρ of N qubits if we know all Pauli expectation values $c_{\alpha_1,\dots,\alpha_N} = \mathrm{tr}[\rho\,(\sigma_{\alpha_1}\otimes\cdots\otimes\sigma_{\alpha_N}]$, with $\alpha = 0,\dots,3$, where $\sigma_0 = \mathbb{1}$, $\sigma_1 = \sigma_x$, etc. How would one reconstruct ρ from the data? Discuss how one would measure the $c_{\alpha_1,\dots,\alpha_N}$ for N ions in a linear trap.

[Note: In practice, the $c_{\alpha_1,\dots,\alpha_N}$ will be noisy and might not even describe a physical state (i.e., the reconstructed ρ can have negative eigenvalues), especially if the measured ρ is close to a pure state. Thus, in the presence of noise more sophisticated ways to infer the state ρ are required.]