Lecture "Quantum Optics" — Exercise Sheet #11

Problem 1 (short, a bit tricky)

In this problem, we will derive a qualitative argument for the magnitude of the enhancement of the spontaneous emission rate for a two-level system placed into a damped cavity, i.e., the Purcell effect, which will allow to reproduce the correct result up to a constant factor.

1. According to Fermi's golden rule, the emission rate into a continuum of modes is proportional to the density of modes, i.e., the number of modes per energy (or rather frequency) and and unit volume, which we will determine in the following for both the vacuum and the damped cavity.
2. To determine the density of modes in the vacuum, first consider a finite volume $W = L \times L \times L$. There, the modes are labelled by integers $\vec{n} = (n_x, n_y, n_z) \in \mathbb{Z}^3$; the wavevector $\vec{k}$ is then given by $\vec{k} = \frac{2\pi}{L}\vec{n}$. Then, each mode takes a k-space volume $(\frac{2\pi}{L})^3$ (up to a constant factor coming from degeneracies, e.g. polarization). We now assume $\vec{k}$ to be continuous, which is justified for large L (we are interested in the limit $L \to \infty$). The k-space volume taken by the modes around a given frequency ω, and thus a given $k = |\vec{k}| = \omega/c$, is then given by the volume of a spherical shell with radius k and width $\mathrm{d}k$. This allows us to determine $\mathrm{d}k$ for a single mode, and thus finally the $\mathrm{d}\omega$ for a single mode. From there, we can derive the density of modes, i.e., the number of modes per $d\omega$ and volume, around a given ω.
3. In order to determine the density of modes for the cavity, we use that the cavity supports only one mode. Due to the finite lifetime of the cavity, the mode is smeared out in frequency space by approximately $\kappa = \omega/Q$, and it occupies the mode volume V of the cavity.
4. Taking the ratio of the above densities of modes, one obtains a qualitative expression for the enhancement of the emission rate in a lossy cavity. Compare this to the formula derived in the lecture for the Purcell effect.

Problem 2 (lengthy)

The aim of this problem is to derive the master equation for optical pumping, and numerically study its solution.

1. Consider a three-level system with the two ground states $|+\rangle$ and $|-\rangle$ and the excited state $|e\rangle$. $|e\rangle$ decays to both $|+\rangle$ and $|-\rangle$, each at rate κ; moreover, a laser couples $|+\rangle$ resonantly to $|e\rangle$ with coupling constant Ω_-. Write down the master equation for the density matrix of the system (i.e., give the Hamiltonian and Lindblad operators.)
2. Use the master equation to obtain a differential equation for the matrix elements of the density matrix ρ. (It can be convenient to do this using a computer algebra system, starting from Hamiltonian and Lindblad operators.)
3. We denote the matrix elements of ρ by
$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{e+} & \rho_{e-} \\ \rho_{+e} & \rho_{++} & \rho_{+-} \\ \rho_{-e} & \rho_{-+} & \rho_{--} \end{pmatrix} .$$
Show that the matrix elements $\rho_{e+} = \rho_{+e}^*$ and $\rho_{e-} = \rho_{-e}^*$ do not couple to the other matrix elements. Since we are interested in a system which is initially in an equal weight mixture $\rho(t = 0) = \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|)$, we can thus neglect those matrix elements.
4. Give the differential equations for the remaining matrix elements ρ_{ee}, ρ_{++}, and $\rho_{e+} + \rho_{+e}$ (note that ρ_{--} is fixed by $\rho_{ee} + \rho_{++} + \rho_{--} = 1$),
$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \rho_{ee} \\ \rho_{++} \\ \rho_{e+} + \rho_{+e} \end{pmatrix} = M \begin{pmatrix} \rho_{ee} \\ \rho_{++} \\ \rho_{e+} + \rho_{+e} \end{pmatrix} . \tag{1}$$

5. Unfortunately, M cannot be diagonalized easily. Thus, we need to resort to the numerical study of the time evolution: Use a computer algebra system to plot the solutions of Eq. (1),

$$\begin{pmatrix} \rho_{ee}(t) \\ \rho_{++}(t) \\ \rho_{e+}(t) + \rho{+}e(t) \end{pmatrix} = \exp(Mt) \begin{pmatrix} \rho_{ee}(t=0) \\ \rho_{++}(t=0) \\ \rho_{e+}(t=0) + \rho{+}e(t=0) \end{pmatrix} ,$$

where we are particularly interested in the case $\rho_{ee}(t=0) = \rho_{e+}(t=0) + \rho{+}e(t=0) = 0$ and $\rho_{++}(t=0) = \frac{1}{2}$. Confirm that $\rho_{--} = 1 - \rho_{ee} - \rho_{++}$ goes to one exponentially. Consider both the case where $\kappa \ll \Omega$ and $\kappa > \Omega$. Show that in the former case, one can observe Rabi oscillations between $|e\rangle$ and $|+\rangle$.

Now consider (still numerically) the speed of the exponential convergence of ρ_{--} to 1. Given that κ is fixed (this is the spontaneous decay rate), how does the convergence rate change if we tune Ω? Which Ω should we therefore prefer? What is the fastest decay rate we can obtain (in terms of κ)?