Lecture "Quantum Optics" — Exercise Sheet #10

Problem 1 (easy)

This problem fills in the missing steps in the analysis of a cavity at frequency $\omega_c/2\pi$ coupled to a thermal environment at temperature T. In addition to the steps mentioned below, you are very much encouraged to go through all the steps carefully!

[*Note:* At the beginning of the derivation given in the lecture, κ_+ and κ_- had been mistakenly swapped twice at the bottom of pg. 79; this is fixed in the newly uploaded version. The general thermodynamic argument for their ratio goes as follows: Emission/absorption of a photon to/from the environment is accompanied by the opposite transition of the environment between an arbitrary pair of energy levels E and E' of the environment which differ by $E' - E = \hbar\omega_c$. The probability for emission/absorption are thus proportional to finding the environment in states E and E', respectively; their ratio is thus $\kappa_-/\kappa_+ = E/E' = \exp(\hbar\omega_c/k_BT)$. In particular, the emission rate κ_- is larger than the absorption rate κ_+. This is independent of the distribution of such pairs of levels in the spectrum of the environment.]

1. Verify that for a single bosonic mode in a thermal state, i.e., where the probability $q(n)$ of having n bosons is proportional to $w(n) = \exp(-\beta n\hbar\omega)$, $\beta = 1/k_BT$, the average number $\langle n\rangle = \sum nq(n)$ of bosons is
$$n_{\text{th}} = \frac{1}{\exp(\hbar\omega/k_BT) - 1} \ .$$
(*Hint:* It can be helpful to use $\langle n\rangle = (-\frac{1}{\hbar\omega}\frac{\mathrm{d}}{\mathrm{d}\beta}Z)/Z = -\frac{1}{\hbar\omega}\frac{\mathrm{d}}{\mathrm{d}\beta}\log Z$, where $Z = \sum w(n)$ is the *partition function*; $-\beta\log Z$ is the *free energy*.)

2. Using the differential equation for the diagonal elements of ρ, $p(n) = \rho_{n,n}$, derived in the lecture,
$$\dot{p}(n) = \kappa(n_{\text{th}} + 1)(n + 1)p(n + 1) + \kappa n_{\text{th}} n p(n - 1) - \kappa\big((n_{\text{th}} + 1)n + n_{\text{th}}(n + 1)\big)p(n) \ ,$$
show that the average photon number in the cavity $\langle n\rangle = \sum np(n)$ evolves according to
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle n\rangle = -\kappa(\langle n\rangle - n_{\text{th}}) \ .$$

Problem 2 (medium)

Derive the master equation for a two-level atom interacting with a thermal environment, following the derivation of a cavity interacting with a thermal environment.

- Argue that the Lindblad operators are of the form $L_\pm = \sqrt{\Gamma_{pm}}\sigma^\pm$, and use the same thermodynamic arguments to express $\Gamma_\pm$ in terms of Γ and n_{th}.
- Give the master equation.
- Use the same arguments as in the lecture to derive the fixed point solution and the convergence rate of the average excitation $\langle e|\rho|e\rangle$. What is average excitation at the fixed point? Does this make sense thermodynamically?
- Find the general solution to the differential equation. How do the different matrix elements decay; in particular, what happens to the off-diagonal elements?

Problem 3 (medium)

Use a computer algebra system to study the solution of the differential equation describing the Purcell effect more closely, taking into account not only the eigenvalues but also the eigenvectors. In particular, investigate the $\kappa \gg \Omega_0$ regime. Given that the system is initially in the state $|e, 0\rangle$, how do the populations of the individual levels evolve? In particular, is there ever a significant photon population in the cavity? Try to determine the leading order of this population, by expanding the eigenvectors in Ω/κ.