Lecture "Quantum Optics" — Exercise Sheet #9

Problem 1 (medium)

Show that any trace preserving completely positive map $T(\rho) = \sum_i M_i \rho M_i^\dagger$ (with $\sum_i M_i^\dagger M_i = \mathbb{1}$) can be realized by adding an ancilla B in a pure state, acting with a unitary on the joint state, and tracing out the ancilla, i.e.,

$$T(\rho_A) = \mathrm{tr}_B \left[U_{AB} (\rho_A \otimes |0\rangle_B \langle 0|_B) U_{AB}^\dagger \right] .$$

Show that the very same way, we can also realize a POVM measurement with POVM operators M_i, by measuring B instead of tracing it out.

Problem 2 (medium)

Consider the transpose channel $T(\rho) = \rho^T$, where ρ^T is the transpose of ρ. [Note that in a ket-bra basis, the transpose acts as $(|i\rangle\langle j|)^T = |j\rangle\langle i|$].

1. Show that T is positive, i.e., it maps positive operators to positive operators, $T(\rho) \geq 0$ for $\rho \geq 0$.
2. Now consider an entangled state $|\Phi\rangle_{AB} = (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)/\sqrt{2}$, and apply T to subsystem A (while acting with the identity channel $I(\rho) = \rho$ on subsystem B). Show that the result $(T_A \otimes I_B)(|\Phi\rangle\langle\Phi|)$ is *not* positive, and thus not a physical state. This implies that the transpose channel T is not *completely positive*, i.e., not a physical map.
3. Show that for any *separable state* $\rho_{AB} = \sum_i p_i \, \rho_{A,i} \otimes \rho_{B,i}$ (with $\rho_{A,i}$ and $\rho_{B,i}$ physical states, i.e., trace = 1 and positive), $(T \otimes I)(\rho_{AB})$ is positive. This implies that $T \otimes I$ (known as the *partial transposition* operation) can be used to detect entangled mixed states; this is the so-called *PPT (positive partial transpose) criterion.*

Problem 3 (tricky)

In this problem, we compute the rate of spontanous decay of an atom in the vacuum. We consider a (large) volume $V = L^3$, and an atom which couples via the Jaynes-Cummings Hamiltonian

$$H = \sum_{\boldsymbol{k},\alpha} i\hbar g_{\boldsymbol{k},\alpha} (\sigma^- a^\dagger - \text{h.c.})$$

to all the modes in the volume. Here, α denotes the two polarizations with wave vector $\boldsymbol{k}$, and

$$g_{\boldsymbol{k},\alpha} = \frac{\Omega_{\boldsymbol{k},\alpha}}{2} = \sqrt{\frac{\omega_{\boldsymbol{k}}}{2\hbar\varepsilon_0 V}} \, (\boldsymbol{d} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\alpha}) \ ,$$

where $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$, $\boldsymbol{d}$ is the dipole moment associated with the transition, and $\boldsymbol{\epsilon}_{\boldsymbol{k},\alpha}$ is a unit vector along the polarization α.

1. We consider the atom initially in the excited state $|e\rangle$, and the light field in the vacuum. Fermi's golden rule states that the transition rate from $|e\rangle$ to $|g\rangle$ is given by

$$\Gamma = \frac{2\pi}{\hbar^2} \sum_{\boldsymbol{k},\alpha} \left| \langle g, 1_{\boldsymbol{k},\alpha} | H | e, 0 \rangle \right|^2 \delta(\omega_{eg} - c|\boldsymbol{k}|) \ ,$$

where ω_{eg} is the transition frequency of the atom, and $\langle g, 1_{\boldsymbol{k},\alpha}|$ is the state with the atom in the ground state and one photon in mode $(\boldsymbol{k}, \alpha)$. Show that from this, it follows that

$$\Gamma = 2\pi \sum_{\boldsymbol{k},\alpha} |g_{\boldsymbol{k},\alpha}|^2 \delta(\omega_{eg} - c|\boldsymbol{k}|) \ .$$

2. Next, we replace the sum over $\boldsymbol{k}$ by an integral (keeping the sum over α). Show that this is done by replacing

$$\sum_{\boldsymbol{k}} \rightarrow \frac{V}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k} \ ,$$

and express Γ as an integral in spherical coordinates $\boldsymbol{k} = (k \sin\theta \cos\phi, k \sin\theta \sin\phi, k \cos\theta)$.

3. Next, we remove the sum over α. To this end, assume that the atomic dipole moment $\boldsymbol{d}$ is alinged along the z axis, $\boldsymbol{d} = (0, 0, d)$. (This is without loss of generality, since the integral is rotationally invariant.) Show that

$$\sum_{\alpha} (\boldsymbol{d} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},\alpha})^2 = d^2 \sin^2\theta \ .$$

(Observe that one can always choose the polarizations α such that one of them is orthogonal to the plane spanned by $\boldsymbol{k}$ and the z axis.)

4. Next, solve the integral over ϕ and θ. (Note: $\int \mathrm{d}\theta \sin^3\theta = \frac{4}{3}$.)

5. Finally, integrate over k to obtain the decay rate Γ. You should find

$$\Gamma = \frac{\omega_{eg}^3 d^2}{3\pi\varepsilon_0 \hbar c^3} \ .$$

Congratulations!