Lecture "Quantum Optics" — Exercise Sheet #8

Problem 1 (easy)

Show that a density matrix ρ describes a pure state $|\psi\rangle\langle\psi|$ if and only if $\mathrm{tr}(\rho^2) = 1$. [$\mathrm{tr}(\rho^2)$ is also known as the *purity* of ρ.]

Problem 2 (medium)

The goal of this problem is to relate different ensemble decompositions of a given mixed state ρ.

- Consider $\rho = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$. Show that ρ can also be written as a mixture of two non-orthogonal states $|\phi_\pm\rangle = \alpha|0\rangle \pm \beta|1\rangle$ for appropriately chosen α, β, and mixing weights, i.e., $\rho = p_+|\phi_+\rangle\langle\phi_+| + p_-|\phi_-\rangle\langle\phi_-|$. Try to classify all decompositions of ρ into two pure states.
- Given two (generally non-orthogonal!) decompositions of the same mixed state, $\rho = \sum_{i=1}^N p_i|\phi_i\rangle\langle\phi_i|$ and $\rho = \sum_{j=1}^M q_j|\psi_j\rangle\langle\psi_j|$, show that any two such decompositions can always be related by a transformation $\sqrt{p_i}|\phi_i\rangle = \sum_{ij} v_{ij}\sqrt{q_j}|\psi_j\rangle$, where v_{ij} is an isometry, i.e., $\sum_{j=1}^M v_{ij}v_{kj}^* = \delta_{ik}$ if $M \geq N$, and otherwise $\sum_{i=1}^N v_{ij}v_{ik}^* = \delta_{jk}$. (Note: It can be helpful to first consider the case where one of the two decompositions is the eigenvalue decomposition, i.e., the vectors $|\phi_i\rangle$ are orthogonal.)

Problem 3 (easy)

In the lecture and exercise sheet 6 (problem 1), we have discussed the Bloch sphere representation of pure states of two-level systems, $|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{1} + \vec{n}\cdot\vec{\sigma})$, where $|\vec{n}| = 1$, $\vec{n} \in \mathbb{R}^3$, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$; which implies that pure states correspond to vectors $\vec{n}$ on the unit sphere.
In this problem, we will extend the Bloch sphere picture to mixed states of two-level systems.

- Show that any density matrix of a two-level system (i.e., any $\rho \geq 0$ with $\mathrm{tr}\,\rho = 1$) can be written as
$$\rho = \tfrac{1}{2}(\mathbb{1} + \vec{n}\cdot\vec{\sigma}) ,$$
where $|\vec{n}| \leq 1$ ($\vec{n} \in \mathbb{R}^3$), i.e., ρ is represented by a point inside the unit sphere.
- Show that ρ is a pure state if and only if $|\vec{n}| = 1$.
- Find the Bloch sphere representation of (i) $\rho = \frac{1}{2}\mathbb{1}$, (ii) $\rho = |0\rangle\langle 0|$ and (iii) $\rho = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$.
- Show that the point on the Bloch sphere representing a convex combination of two mixed states, $\rho = p_1\rho_1 + p_2\rho_2$ ($p_1 > 0$, $p_2 > 0$, $p_1 + p_2 = 1$), is given by $\vec{n} = p_1\vec{n}_1 + p_2\vec{n}_2$ – this is, a convex combination of states corresponds to a convex combination of the corresponding points in the Bloch sphere.

Problem 4 (easy)

Compute the reduced density matrices $\rho_A = \mathrm{tr}_B(\rho)$ and $\rho_B = \mathrm{tr}_A(\rho)$ for the following states $\rho \equiv \rho_{AB}$ of two two-level systems (qubits) A and B:

- $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = |a\rangle_A|b\rangle_B$.
- $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = (|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}$.
- $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = (|01\rangle_{AB} - |10\rangle_{AB})/\sqrt{2}$.
- A general two-qubit state
$$\rho = \begin{pmatrix} a & b & c & d \\ b^* & e & f & g \\ c^* & f^* & h & k \\ d^* & g^* & k^* & l \end{pmatrix} ,$$
where the ρ is written in the basis $\{|00\rangle_{AB}, |01\rangle_{AB}, |10\rangle_{AB}, |11\rangle_{AB}\}$.