Lecture "Quantum Optics" — Exercise Sheet #6

Problem 1 (part 1+2 easy, part 3 tricky)

1. Consider a two-level atom which is initially prepared in state $|e\rangle$, and which interacts resonantly (i.e., the detuning is zero, $\Delta_r = 0$) with a general single-mode light field $|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, as described by the Jaynes-Cummings-model. What is the state of the system (atom+light field) at time t?

2. Now let $|\psi\rangle$ be a coherent light field. What is the probability $p(t)$ to find the atom in the excited state after an interaction time t? Study the resulting function $p(t)$ numerically for different strengths $|\alpha|^2$ of the coherent field (both for weak and strong fields). What do you find? How does it compare to the behavior which we have found for a classical light field of the same strength?

3. Can you (qualitatively) explain the behavior observed? How can one estimate the time scales appearing? (For this, remember that a for a coherent beam, the average photon number is $n_0 = |\alpha|^2$, and the standard deviation is $\sqrt{n_0}$. For $|n - n_0| \ll n_0$, you can approximate $\sqrt{n+1} \approx \sqrt{n_0+1}(1 + \frac{1}{2}\frac{n-n_0}{n_0+1})$.

Problem 2 (medium)
In problem 3, it is shown how to derive an effective Hamiltonian for the interaction between an atom and a light mode in the so-called dispersive regime, i.e., when the detuning $\Delta \gg \Omega_0$, which is of the form

$$H_{\text{disp}} = \frac{\Omega_0^2 \hbar}{4\Delta} \left(\sigma^+\sigma^- + \sigma_z a^\dagger a\right)$$

In this problem, we want to compare the action of this Hamiltonian with the resonant ($\Delta = 0$) Jaynes-Cummings Hamiltonian

$$H_{JC} = -\frac{i\Omega_0\hbar}{2}(\sigma^+ a - \sigma^- a^\dagger) \ .$$

- Consider the atom initially in the state $|e\rangle$, and the light in the Fock state $|n\rangle$. Compare what happens if they interact via (i) H_{JC} and (ii) H_{disp}. Show that the state evolving under H_{disp} is a product state at all times, while H_{JC} entangles the atom with the cavity field. Check that the system also stays in a product state under H_{disp} if the atom is initially in the state $(|e\rangle + |g\rangle)/\sqrt{2}$.

- Consider the case where the atom is initially in the state $(|e\rangle + |g\rangle)/\sqrt{2}$, and the light field is in a coherent state $|\alpha\rangle$.
 (i) Let them interact via H_{disp}. What is the state after a time t? In particular, what is the state after a time $\Omega_0^2 t/4\Delta = \pi/2$? Can the two states of the light field which occury be distinguished classically for sufficiently large $|\alpha|$, e.g. with an interference experiment (i.e., are they orthogonal)? If we measure the atom in the basis $(|e\rangle \pm i|g\rangle)/\sqrt{2}$, what is the state of the light after the measurement? (*Note:* Such macroscopically distinguishable superpositions are called "Schrödinger cats".)
 (ii) Using the intuition built in Problem 1 (and possibly using some numerics) compare this to what happens for H_{JC} for strong light fields if the system interacts for a time of order unity, i.e., $\Omega_0 t \sim 1$.

Problem 3 (lengthy)
In this problem, we study the effective evolution of an atom in a strongly detuned cavity, $\Delta_r \gg \Omega_0$, the so-called *dispersive regime.* To this end, consider the Hamiltonian $H = H_0 + V$, with $H_0 = \hbar\omega_r(a^\dagger a + \frac{1}{2} + \frac{1}{2}\sigma_z) + \frac{1}{2}\hbar\Delta_r\sigma_z$ and $V = -\frac{i}{2}\hbar\Omega_0(\sigma^+ a - \sigma^- a^\dagger)$.

- First, transform the Hamiltonian into an interaction picture w.r.t. H_0, this is, the state of the system evolves as

$$-i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = V_I(t)|\psi\rangle \ ,$$

with $V_I(t) = e^{iH_0t/\hbar} V e^{-iH_0t/\hbar}$. Determine the explicit form of $V_I(t)$. (It should look similar to V, but with time-dependent phases for each of the two terms.)

The time evolution of $|\psi(t)\rangle$ is formally described by the time-ordered exponential

$$|\psi(t)\rangle = \mathcal{T}\left[\exp\left(-i\int_0^t \mathrm{d}t' V(t')/\hbar\right)\right]|\psi(0)\rangle \ .$$

This can be expanded to second order as

$$|\psi(t)\rangle = \left[1 - \frac{i}{\hbar}\int_0^t \mathrm{d}t' V(t') - \frac{1}{\hbar^2}\int_0^t \mathrm{d}t' V(t') \int_0^{t'} \mathrm{d}t'' V(t'')\right]|\psi(0)\rangle \ . \tag{1}$$

- Determine the first- and second-order integrals in Eq. (1). In the second-order integral, drop all terms which scale like Ω_0^2/Δ^2 (this can be used to argue that most terms can be dropped already before carrying out the outer integral).

You should now be left with

$$|\psi(t)\rangle = \left[1 + \frac{i\Omega_0}{2\Delta}\left(\left(e^{i\Delta t} - 1\right)\sigma^+ a + \text{h.c.}\right) + \frac{\Omega_0^2 t}{4i\Delta}\left(\sigma^+\sigma^- + \sigma_z a^\dagger a\right)\right]|\psi(0)\rangle \ .$$

- Estimate the time scales on which the first term and on which the second term evolves, and show that for long enough times $t \simeq \Delta/\Omega_0^2$, the second term dominates (while the first averages out).
- By taking the time derivative of the second term, you can now derive the effective Hamiltonian

$$H_{\text{eff}} = \frac{\Omega_0^2\hbar}{4\Delta}\left(\sigma^+\sigma^- + \sigma_z a^\dagger a\right)$$

for the evolution in the dispersive regime. (Note that this Hamiltonian is very different from the Jaynes-Cummings model: It does not change the state of the system in the "canonical" basis, but only introduces phase shifts between them. As we see in Problem 2, this can still be used to entangle the atom with the light.)