Lecture "Quantum Optics" — Exercise Sheet #5

Problem 1 (easy)

1. Any pure state of a single qubit (i.e., a two-level system) can be written as
$$|\psi\rangle = e^{i\alpha}\big[\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\big]\ . \tag{1}$$
Show that this implies that[1]
$$|\psi\rangle\langle\psi| = \tfrac{1}{2}(\mathbb{1} + \vec{v}\cdot\vec{\sigma}) \quad \text{with } \vec{v}\in\mathbb{R}^3 \text{ and } |\vec{v}| = 1\ , \tag{2}$$
(i.e., $\vec{v}$ is a vector on the unit sphere in $\mathbb{R}^3$), where $\vec{v}\cdot\vec{\sigma} = \sum_{i=1}^3 v_i\sigma_i$ with $\sigma_1 \equiv \sigma_x$, $\sigma_2 \equiv \sigma_y$, and $\sigma_3 \equiv \sigma_z$, and determine the form of $\vec{v}$. (You should find that $\vec{v}$ is expressed in spherical coordinates in θ and ϕ.)

   Further, show that conversely, any state of the form of the r.h.s. of Eq. (2) is indeed a pure qubit state, i.e., of the form (1).

   Note: The vector $\vec{v}$ is called the *Bloch sphere representation* of the state $|\psi\rangle$ (see Lecture 5).

2. Show that the expectation value of the Pauli operators is $\langle\psi|\sigma_i|\psi\rangle = v_i$; i.e., $|\psi\rangle$ desribes a spin which is polarized along the direction $\vec{v}$. (*Note:* This is particularly easy if you use that $\langle\psi|O|\psi\rangle = \mathrm{tr}[|\psi\rangle\langle\psi|\,O]$ and $\mathrm{tr}[\sigma_i\sigma_j] = 2\delta_{ij}$.)

3. Show that for any state $|\psi\rangle$ with corresponding Bloch vector $\vec{v}$, the state $|\phi\rangle$ orthogonal to it, i.e. with $\langle\psi|\phi\rangle = 0$ (for qubits, i.e., in $\mathbb{C}^2$, this state is uniquely determined up to a phase!), is described by the Bloch vector $-\vec{v}$, i.e., it is located at the opposite point of the Bloch sphere.

4. Let $H = c\mathbb{1} + \vec{w}\cdot\vec{\sigma}$ with $\vec{w}\in\mathbb{R}^2$. What are the eigenvectors and eigenvalues of H? In particular, which are the Bloch vectors describing the eigenvectors? (*Note:* Try to use Eq. (2).)

Problem 2 (medium)

1. Let $x\in\mathbb{R}$, and A a matrix such that $A^2 = \mathbb{1}$. Show that $\exp(iAx) = \cos(x)\mathbb{1} + i\sin(x)A$.

2. Let $\vec{n}\in\mathbb{R}^3$ such that $|\vec{n}| = 1$, and define
$$R_{\vec{n}}(\theta) = \exp(-i\theta\vec{n}\cdot\vec{\sigma}/2)\ .$$
Using part 1, show that
$$R_{\vec{n}}(\theta) = \cos(\theta/2)\,\mathbb{1} - i\sin(\theta/2)(\vec{n}\cdot\vec{\sigma})\ .$$

3. Show that $R_{\vec{n}}(\theta)$ is unitary, and that conversely any one-qubit unitary operator U is of the form
$$U = e^{i\alpha}R_{\vec{n}}(\theta)\ .$$
Note: It can help to use that $\{\mathbb{1},\sigma_1,\sigma_2,\sigma_3\}$ forms a basis (over $\mathbb{C}$) for the space of *all* complex 2×2 matrices. (If you use this, you should convince yourself that this is true!)

Problem 3 (tricky)

Show that $R_{\vec{n}}(\theta)|\psi\rangle$ rotates the Bloch vector $\vec{v}$ which describes $|\psi\rangle$ by an angle θ about the axis $\vec{n}$.

Hint: It is best to study how $|\psi\rangle\langle\psi|$ transforms, using the representation (2). Start by proving the claim for rotations about the z axis, i.e., $\vec{n} = (0,0,1)$. To generalize this to rotations about arbitrary axes, it is most convenient define a "rotated Pauli basis" $\vec{\sigma}' = (\sigma_1',\sigma_2',\sigma_3')$ with $\sigma_3' = \vec{n}\cdot\vec{\sigma}$ and σ_1', σ_2' chosen such that they satisfy the Pauli commutation relations, and re-express $\vec{n}\cdot\vec{\sigma} = \vec{n}'\cdot\vec{\sigma}'$ and $\vec{v}\cdot\vec{\sigma} = \vec{v}'\cdot\vec{\sigma}'$ in the $\vec{\sigma}'$ basis. This corresponds to choosing a new basis for the Bloch sphere such that $\vec{n}$ points along the z axis. You can then convince yourself that because the new Pauli basis $\vec{\sigma}'$ satisfies the same algebraic relations, this describes a rotation about the re-defined z axis.
Alternatively, you can work out the full 3×3 transformation matrix for $\vec{v}$ and compare it to the transformation matrix of a general rotation in $\mathbb{R}^3$.

---

[1] *Note:* The normalization in Eq. (2) given in the lecture was incorrect; it has been fixed in the uploaded lecture notes.