Lecture "Quantum Optics" — Exercise Sheet #3

Problem 1 (part 1 easy, part 2 medium)

- Verify that for two coherent states $|\alpha\rangle$ and $|\beta\rangle$, it holds that

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\alpha^*\beta}\ , \quad \text{and} \quad |\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}\ .$$

- Show the completeness relation

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha|\alpha\rangle\langle\alpha| = \mathbb{1}\ .$$

  (*Hint:* Use the Fock basis representation of the coherent state, and use polar coordinates for α.)

Problem 2 (medium, some tricky parts)

1. Show that the Hamiltonian $H_{ab} = \hbar(ia^\dagger b - ib^\dagger a)$ generates the beam splitter transformation, i.e., $U(t) = \exp(-iH_{ab}t/\hbar)$ transforms a and b as

$$U(t)^\dagger\, a\, U(t) = \cos(t)\, a + \sin(t)\, b$$
$$U(t)^\dagger\, b\, U(t) = \cos(t)\, b - \sin(t)\, a\ .$$

   (*Note:* In the lecture, the order of U and $U^\dagger$ was reversed, this has been fixed in the uploaded lecture notes.)

2. Show that the generalized beam splitter Hamiltonian $H_{ab}(\theta) = \hbar(e^{i\theta}a^\dagger b + e^{-i\theta}b^\dagger a)$ realizes a beam splitter transformation $U_\theta(t) = \exp(-iH_{ab}(\theta)t/\hbar)$ with an arbitrary phase shift between the modes a and b. (*Note:* Based on part 1 of the problem, this can be done without doing any calculation.)

3. Show that a beam splitter acts trivially on the vacuum, $U(t)|0,0\rangle = |0,0\rangle$.

Problem 3 (easy)

For any analytic function $f(x) = \sum_{n=0}^\infty c_n x^n$, we can define the action of f on an operator A via $f(A) := \sum_{n=0}^\infty A^n$. Show that for any unitary transformation U, $U\, f(A)\, U^\dagger = f(UAU^\dagger)$ (and, more generally, for any invertible X, $X\, f(A)\, X^{-1} = f(XAX^{-1})$). (*Note:* This implies that the action of functions on operators can be understood as applying the function to its eigenvalues.)