Lecture "Quantum Optics" — Exercise Sheet #2

Problem 1 (easy)

Prove the following relations for the *displacement operator* $D = \exp(\alpha a^\dagger - \alpha^* a)$ (using Problem 4 from Sheet 1):

- $D(\alpha) = e^{-i\,\mathrm{Re}\,\alpha\,\mathrm{Im}\,\alpha} e^{i(\sqrt{2}\mathrm{Im}\,\alpha)Q} e^{-i(\sqrt{2}\mathrm{Im}\,\alpha)P}$, where $Q = (a^\dagger + a)/\sqrt{2}$ and $P = i(a^\dagger - a)/\sqrt{2}$ are the position and momentum operator. (*Note:* This is consistent with the fact that the operator $\exp(-iqP)$ generates a translation by q in position space, and the operator $\exp(ipQ)$ a translation by p in momentum space, as you might know from quantum mechanics.)
- $D(\alpha+\beta) = e^{-i\,\mathrm{Im}\,(\alpha\beta^*)} D(\alpha) D(\beta)$

Problem 2 (medium)

Let A and B be hermitian operators. Prove the uncertainty relation

$$\Delta A \Delta B \geq \tfrac{1}{2} |\langle [A,B] \rangle| \ ,$$

where $\langle O \rangle \equiv \langle \psi | O | \psi \rangle$ (for a given state $|\psi\rangle$), and the variance $\Delta O \geq 0$ is defined via $(\Delta O)^2 = \langle (O - \langle O \rangle)^2 \rangle = \langle O^2 \rangle - (\langle O \rangle)^2$.

Hint: Consider first the case $\langle A \rangle = \langle B \rangle = 0$, and use the Cauchy-Schwarz inequality to bound $|\langle AB \rangle|$.

Problem 3 (medium)

Show directly that the solution of the eigenvalue equation $a|\alpha\rangle = \alpha|\alpha\rangle$ is given by

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \ .$$

Hint: Make an ansatz $|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$. From $a|\alpha\rangle = \alpha|\alpha\rangle$, you can derive a recursion relation for the c_n, which determines $|\alpha\rangle$ up to normalization.

Problem 4 (tricky)

Consider the *squeezing operator* $S(\varepsilon) = \exp\left[\frac{\varepsilon^*}{2} a^2 - \frac{\varepsilon}{2} (a^\dagger)^2\right]$, where $\varepsilon = re^{2i\phi}$, with $r > 0$. Prove the following relations:

- $S^\dagger(\varepsilon) = S(-\epsilon) = S^{-1}(\varepsilon)$, i.e., $S(\varepsilon)$ is unitary.
- $S^\dagger(\varepsilon) a S(\varepsilon) = a\cosh(r) - a^\dagger e^{2i\phi} \sinh(r)$.
  Hint: Use Exercise 4 from Sheet #1, and show that the nested commutators show an alternating pattern.
- $S^\dagger(\varepsilon) X_\phi S(\varepsilon) = X_\phi e^{-r}$, and $S^\dagger(\varepsilon) X_{\phi+\pi/2} S(\varepsilon) = X_\phi e^{r}$, where $X_\phi = (e^{-i\phi} a + e^{i\phi} a^\dagger)/\sqrt{2}$.