Lecture "Quantum Optics" — Exercise Sheet #1

Problem 1 (easy)
Using only $[a, a^\dagger] = \mathbb{1}$, $\hat{n} := a^\dagger a$, and $\hat{n}|n\rangle = n|n\rangle$, verify that

- $\hat{n}a = a(n-1)$, and thus $a|n\rangle \propto |n-1\rangle$
- $\hat{n}a^\dagger = a(n+1)$, and thus $a^\dagger|n\rangle \propto |n+1\rangle$
- determine the normalizations in $a|n\rangle = C_n|n-1\rangle$ and $a^\dagger|n\rangle = C'_n|n-1\rangle$.
- $aa^\dagger|n\rangle = (n+1)|n\rangle$

Problem 2 (medium)
For $H = \hbar\omega(a^\dagger a + \frac{1}{2})$, verify

$$e^{iHt/\hbar} a e^{-iHt/\hbar} = e^{-i\omega t} a \ ,$$

and determine the corresponding identity for

$$e^{iHt/\hbar} a^\dagger e^{-iHt/\hbar} \ .$$

Hint: Expand H in its eigenbasis, $H = \sum_{n=0}^\infty E_n |n\rangle\langle n|$.

Problem 3 (medium)
Consider a system consisting of two modes with Fock basis $|n_1, n_2\rangle$, and corresponding annihilation operators a_1 acting on mode 1 and a_2 acting on mode 2.

1. What is the most general state $|\psi\rangle$ with two photons in total, i.e., for which $(a_1^\dagger a_1 + a_2^\dagger a_2)|\psi\rangle = 2|\psi\rangle$?
2. Among all two-photon states, find the states which satisfy $a_1|\psi\rangle = a_2|\psi\rangle$, i.e., for which we cannot tell whether a photon has been removed from mode 1 or mode 2.

(*Remark:* Such states appear if we transform e.g. from a basis of counterpropagating waves to a basis of standing waves – if an atom absorbs a photon from a standing wave, it will not get any momentum, and thus, there must be no way to distinguish whether the photon came from a left or a right propagating mode.)

Problem 4 (tricky)
For an operator A, e^A is defined via its Taylor series, $e^A = \sum_{n=0}^\infty \frac{A^n}{n!}$. While for commuting operators A and B, $e^{A+B} = e^A e^B$ (can you prove this from the Taylor series?), this is generally false if $[A, B] \neq 0$. In this problem, we will prove some properties of matrix exponentials.

1. Prove that for any operators A and B and $\chi \in \mathbb{C}$,

$$f(\chi) := e^{\chi A} B e^{-\chi A} = B + \chi[A, B] + \frac{\chi^2}{2!}[A, [A, B]] + \frac{\chi^3}{3!}[A, [A, [A, B]]] + \ldots \tag{1}$$

   Hint: The right hand side of (1) is the Taylor series of $f(\chi)$ around $\chi = 0$, i.e., we need to show that the commutators in the sum are $\frac{\mathrm{d}^k f}{\mathrm{d}\chi^k}|_{\chi=0}$. This can be shown recursively: Show that $f'(\chi) = e^{\chi A}[A, B]e^{-\chi A}$, and use that this is of the same form as $f(\chi)$, with B replaced by $[A, B]$.

2. If two operators A and B satisfy the conditions

$$[A, [A, B]] = [B, [A, B]] = 0 \tag{2}$$

   then

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \ . \tag{3}$$

   This is a special case of the *Baker-Campbell-Hausdorff relations.*

   Hint: Define $f(x) = e^{xA}e^{xB}$, and express the derivative of f in the form $f'(x) = O(x)f(x)$. Use equations (1) and (2) to simplify $O(x)$, and show that this differential equation is satisfied by an exponential $f(x) = e^{Q(x)}$. By setting $x = 1$ and again using (2), you can prove (3).