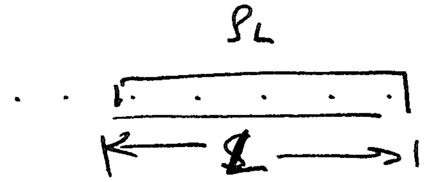
The area law

$H = \sum h_i$, gapped $\Rightarrow$ $|\psi_0\rangle$ G.S.

1D:

$|\psi_0\rangle$ 

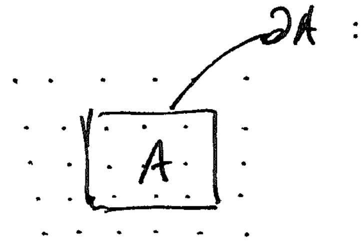
$\Rightarrow S(\rho_L) \leq$ const.

(proven – Hastings)

even for gapless H typ.: $S(\rho_L) \sim \log L$!

(but there exist examples w/ $S(\rho_L) \propto L$!)

2D:



$S(\rho_A) \sim |\partial A|$

(or $|\partial A| \cdot \log |A|$)

$\longrightarrow$ scales like surface area vs. volume:

$$\boxed{\Rightarrow \text{Area law}}$$

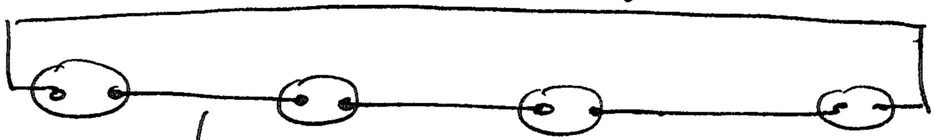
Intuition: Entanglement is concentrated around boundary (for any partition !)

Ansatz for area law (1D):

1. Decompose each site into 2 subsystems:

$\mathbb{C}^D \times \mathbb{C}^D$

2. Put sites into max. entangled states ("bonds")



$$|\omega_D\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i\rangle|i\rangle$$

D ← "bond dimension"

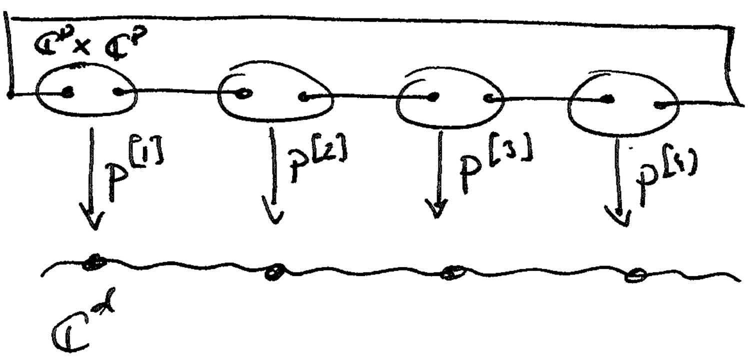
→ This ansatz has exactly an area law behavior:

for <u>any</u> partition, we cut <u>exactly</u> two bonds!

$$\boxed{S(\rho_L) = 2 \log D}$$

This is of course a bit boring: no NNN-correlations, ...

3. Apply a map $P^{[s]} : \mathbb{C}^D \times \mathbb{C}^D \to \mathbb{C}^d$ at each sites:



$$|\Psi\rangle = P^{[1]} \otimes \ldots \otimes P^{[N]} \, |\omega_D\rangle^{\otimes N}$$

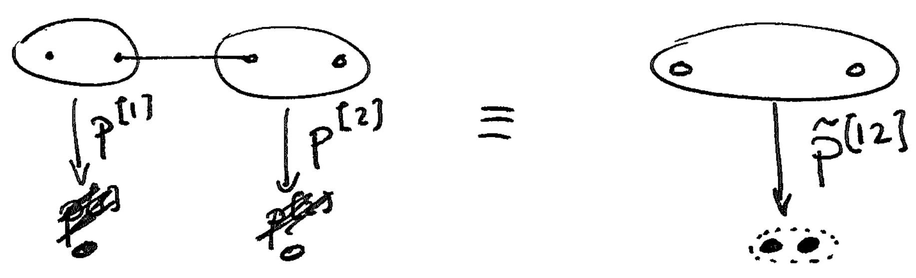
- This gives a <u>family</u> of states which can be enlarged by increasing D.
- <u>$D=1$</u>: <u>Product states</u> are special case.
- parameters: $\underline{N \cdot D^2 \cdot d} \longrightarrow$ efficient scaling in N.
- can approximate states w/ area law (and thus g.s. of gapped Hams.) with <u>moderate</u> scaling of D.

Matrix Product States:

What is the <u>explicit form</u> of $|\Psi\rangle$,

$$|\Psi\rangle = \sum c_{i_1 \ldots i_N} \, |i_1, \ldots, i_N\rangle \; ?$$

Consider:



$$\equiv \Big(\underset{\substack{\uparrow \\ A,B}}{P^{[1]}} \otimes \underset{\substack{\uparrow \\ C,D}}{P^{[2]}} \Big) \cdot \underset{\substack{\uparrow \\ B,C}}{|\omega_0\rangle}$$

Write $P^{[s]} = \sum_{i\alpha\beta} A^{[s],i}_{\alpha\beta} \, |i\rangle\langle\alpha,\beta|$

$$\Big(\underset{AB}{P^{[1]}} \otimes \underset{CD}{P^{[2]}}\Big)|\omega_D\rangle_{BC} = \sum_{i\alpha\beta} A^{[1],i}_{\alpha,\beta} \, |i\rangle\langle\alpha\beta|_{AB} \cdot$$

$$\sum_{j\gamma\delta} A^{[2],j}_{\gamma\delta} \, |j\rangle\langle\gamma,\delta|_{CD} \cdot \sum_{k=1}^{D} |k,k\rangle_{B,C}$$

$$\beta = \gamma$$

$$= \sum_{ij\alpha\beta\delta} A^{[1],i}_{\alpha\beta} A^{[2],j}_{\beta\delta} \, |i,j\rangle\langle\alpha,\delta|$$

$$= \sum_{(ij)\alpha\delta} \left(A^{[1],i} A^{[2],j}\right)_{\alpha\delta} |i,j\rangle\langle\alpha,\delta|$$

$=: \tilde{P}^{[12]} \leftarrow$ Same form as before,

with $A^{[s],i} \longrightarrow A^{[s]i_s} A^{[s+1]\,i_{s+1}}$.

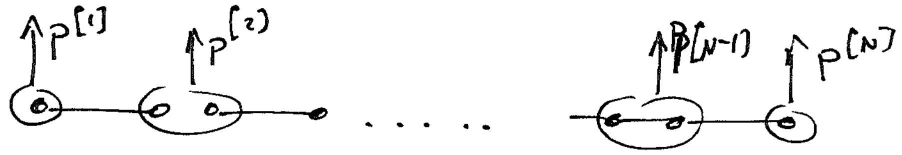
Iterate this argument:

Weight of config $|i_1, \ldots, i_N\rangle$ is

$$c_{i_1 \cdots i_N} = \mathrm{tr}\left[A^{[1],i_1} A^{[2],i_2} \cdots\cdots A^{[N],i_N}\right]$$

$\Longrightarrow$ $\boxed{\text{Matrix Product States}}$

* This can also be done with <u>open</u> boundary conditions (OBC)



$$|\psi\rangle = \sum_{i_1,\dots,i_N} \underset{1\times N}{A^{[1],i_1}} \cdot \underset{N\times N}{A^{[2],i_2}} \cdots \underset{N\times 1}{A^{[N],i_N}} \; |i_1,\dots,i_N\rangle$$

$$\left(= \sum_{i_1,\dots,i_N} \langle e|\, \tilde{A}^{[1],i_1} A^{[2],i_2} \cdots \tilde{A}^{[N],i_N} |r\rangle \; |i_1,\dots,i_N\rangle \right)$$

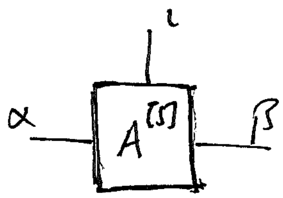
* We can also extend this by allowing for <u>different bond dimensions</u> on each link:

$$\sum_{i_1\dots i_N} \mathrm{tr}\left[\underset{D_1\times D_2}{A^{[1],i_1}} \cdot \underset{D_2\times D_3}{A^{[2],i_2}} \cdots \cdots \underset{D_N\times D_1}{A^{[N],i_N}} \right] |i_1,\dots,i_N\rangle$$

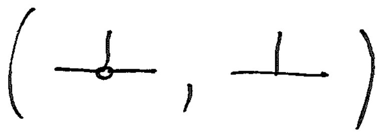
* Tensor network notation:

$$A^{[s]} \equiv A^{[s]\,i}_{\alpha\beta} \quad : \text{tensor w/ 3 indices.}$$

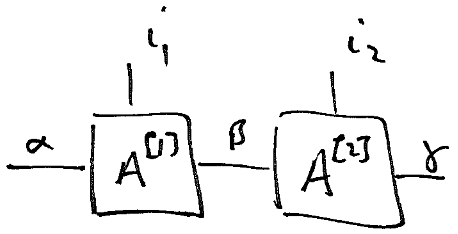
Graphical notation:



$\equiv A^{[s]}$



Summing over index ("contraction") ≡ connect legs:



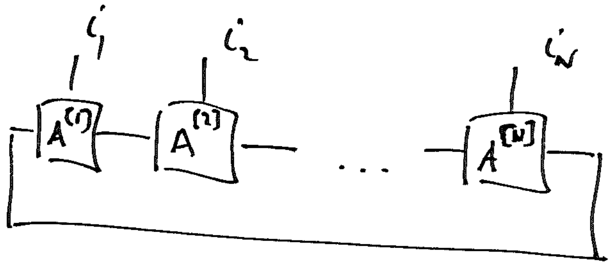
$$\equiv \sum_{\beta} A^{[1],\,i_1}_{\alpha\beta}\, A^{[2],\,i_2}_{\beta\gamma} \equiv \left(A^{[1]i_1}\cdot A^{[2]i_2}\right)_{\alpha\gamma}$$

MPS: $|\psi\rangle = \sum c_{i_1 \dots i_N}\, |i_1, \dots, i_N\rangle$

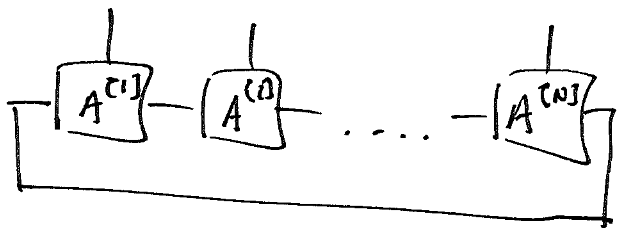
N-index tensor

$$c_{i_1,\dots,i_N} = \mathrm{tr}\left[A^{[1],\,i_1} \dots A^{[N],\,i_N}\right]$$

$=$



We will also write shorthand

$|\psi\rangle =$ 

Some examples:

* GHZ state: $|\psi\rangle = |0\ldots\ldots 0\rangle + |1\ldots\ldots 1\rangle$

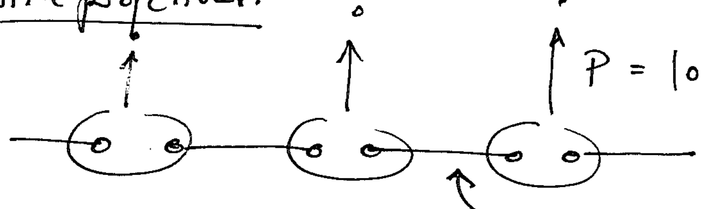
- MPS (D=2): $A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; PBC

(PBC)

$$A^0 A^0 = A^0$$
$$A^0 A^1 = 0$$
$$A^1 A^1 = A^1$$

$$c_{i_1 \ldots i_N} = \mathrm{tr}\left[A^{i_1} \ldots\ldots A^{i_N}\right] = \begin{cases} \mathrm{tr}(A^0) = 1\,; & i_1 = \ldots = i_N = 0 \\ \mathrm{tr}(A^1) = 1\,; & i_1 = \ldots = i_N = 1 \\ 0 & \text{otherwise} \end{cases}$$

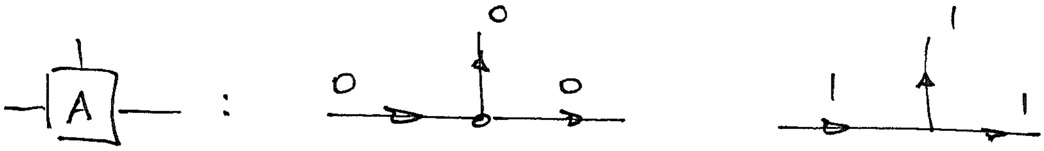
- with projectors:



$P = |0\rangle\langle 0,0| + |1\rangle\langle 1,1|$

$|00\rangle + |11\rangle$

"classical agents" picture:



(interpret tensors as "agents" passing information from left to right, and keeping some local info)

alternatively: OBC:

$$\langle l| = \langle +| \; ; \quad |r\rangle = |+\rangle$$

$$\langle +| A^{i_1} \cdots A^{i_N} |+\rangle = \begin{cases} \frac{1}{2}, & i_1 = i_2 = \ldots = i_N \\ 0 & \text{otherwise} \end{cases}$$

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* W state:

$$|\psi\rangle = |10\ldots0\rangle + |010\ldots\rangle + |0010\ldots\rangle + \ldots$$

MPS: $A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$; $A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 1|$

D=2, OBC

$$\langle l| = \langle 0| \; ; \quad |r\rangle = |1\rangle$$

Note: $A^0 \cdot A^0 = A^0$

$$A^0 \cdot A^1 = A^1 = |0\rangle\langle 1|$$

$$A^1 \cdot A^1 = 0$$

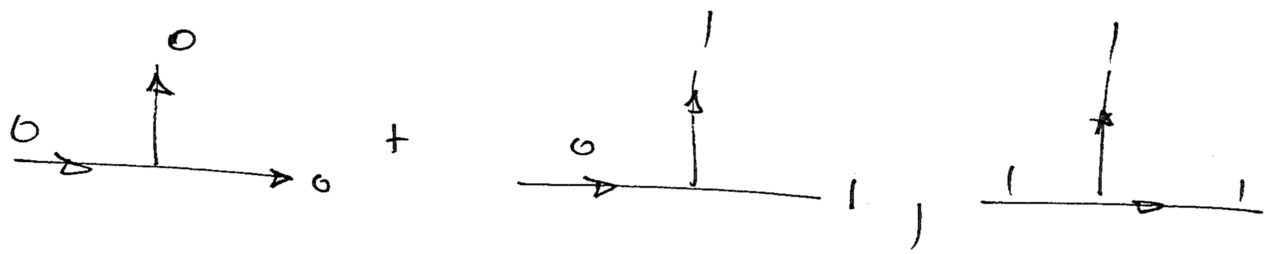
$\Rightarrow$ at most one A^1 in string

$$A^{i_1} \cdot \ldots \cdot A^{i_N} \; !$$

$$A^{i_1} \cdot \ldots \cdot A^{i_N} = \begin{cases} A^0 \; ; & \text{if } i_1 = \ldots = i_N = 0 \\ A^1 \; ; & \text{if } i_1 + i_2 + \ldots + i_N = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \langle 0 | A^{i_1} \cdot \ldots \cdot A^{i_N} | 1 \rangle = \begin{cases} \langle 0 | A^1 | 1 \rangle = 1 \; ; & \sum i_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

"agent" picture:



(pass bit from left to right & output zero, or output 1 and increase bit by one)

— bond space can be understood as counting the # of 1's which were output.

Note: No PBC rep. of W with fixed D known!