Entanglement Theory:

Prelude: The singular value decomposition (SVD)

Theorem: For any matrix M,

$$M = U D V^\dagger;$$ with U, V unitary

and $$D = \begin{pmatrix} s_1 & & & & & \\ & \ddots & & & 0 & \\ & & s_k & & & \\ & 0 & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$ rectangular

$$s_1 \geq s_2 \geq \dots \geq s_k > 0 \quad \text{"singular values"}$$

~~Obs~~ U, V unique up to ~~[illegible]~~ degenerate subspaces.

Observation:

$$M^\dagger M = V D U^\dagger U D^\dagger V^\dagger$$ has eigenvalues s_i^2.

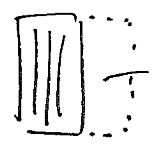
Proof: Consider $M^\dagger M$. Find V s.th.

$$V^\dagger M^\dagger M V = \Lambda \leftarrow$$ diag. & positive. For simplicity, all eigenvalues > 0.

Let $U = M V \Lambda^{-1/2}$. Then:

$$U \sqrt{\Lambda} V^\dagger = M V \Lambda^{-1/2} \Lambda^{+1/2} V^\dagger = M .$$

And: $$U^\dagger U = (\Lambda^{-1/2}) \cdot \underbrace{V^\dagger M^\dagger M V}_{=\Lambda} \Lambda^{-1/2} = \mathbb{1}$$

$\Rightarrow$ U isometry.  complete to unitary. $\tilde{U}$

etc.

$$\tilde{U} \cdot \begin{bmatrix} \sqrt{\Lambda} \\ 0 \end{bmatrix} V = M .$$

Entanglement: bipartite scenario:

Alice $\xleftrightarrow[\text{com.}]{\text{classical}}$ Bob

$\mathbb{C}^d$ — quantum systems — $\mathbb{C}^d$

$|\psi\rangle$ total state:

typ. setting: LOCC (local op. & c. c.)

<u>Product state:</u>

$$|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$$

A's & B's system completely indep.

measurement outcomes can be predicted <u>individually</u>

$\rightarrow$ A measures $|\phi_A\rangle$, B $|\phi_B\rangle$, <u>no corell.</u> betw. outcomes

<u>General state:</u>

$$|\psi\rangle = \sum_{ij} c_{ij} \, |i\rangle \otimes |j\rangle$$

Use <u>SVD</u>:

$$c_{ij} = \sum_k u_{ik} \, s_{k} \, \overline{v}_{jk}$$

$$\Rightarrow |\psi\rangle = \sum_{ijk} u_{ik} \, s_k \, \overline{v}_{jk} \, |i\rangle \otimes |j\rangle$$

$$= \sum_k s_k \, |\alpha_k\rangle \otimes |\beta_k\rangle \, ,$$

wobei $|\alpha_k\rangle = \sum_i u_{ik} |i\rangle$

$|\beta_k\rangle = \sum_j \overline{v_{jk}} |j\rangle$

<u>NB</u>: $\langle \alpha_k | \alpha_{k'} \rangle = \delta_{kk'}$

$\langle \beta_k | \beta_{k'} \rangle = \delta_{kk'}$

Schmidt decomposition

"Schmidt coefficients"

$$\boxed{|\psi\rangle = \sum_k s_k |\alpha_k\rangle |\beta_k\rangle}$$

unique up to degeneracies of the s_k.

<u>Note</u>: i) Any two bip. states w/ same Schmidt spec. can be converted into another by LU & CC.

ii) Any two states which can be interconverted by LU & CC have same Schmidt spectrum!

<u>In particular</u>: Schmidt # $= 1$ $\Longleftrightarrow$ product state

— " — > 1 $\longleftrightarrow$ <u>not</u> prod. state!

"<u>entangled state</u>"

SR <u>cannot be increased</u> by LOCC!

- How can we determine Schmidt coefficients?

- Consider <u>reduced state</u> of A:

Given state ρ_{AB}, $\rho_A = \mathrm{tr}_B\, \rho_{AB} := \sum_i {}_B\langle i|\rho_{AB}|i\rangle_B$

"partial trace".

In part., $\mathrm{tr}[O_A \otimes \mathbb{1}_B\, \rho_{AB}] =$

$$= \sum_{ij} \langle i,j|\, O_A \otimes \mathbb{1}_B\, \rho_{AB}\, |i,j\rangle$$

$$= \sum_{ijkl} \underbrace{\langle ij|\, O_A \mathbb{1}_B\, |k,l\rangle}_{\langle i|O_A|k\rangle\langle j|l\rangle} \langle k,l|\, \rho_{AB}\, |ij\rangle$$

$$= \sum_{ik} \langle i|O_A|k\rangle \underbrace{\sum_j \langle k,j|\, \rho_{AB}\, |i,j\rangle}_{\langle k|\rho_A|i\rangle}$$

$$= \sum_{ik} \langle i|\, O_A \underbrace{|k\rangle\langle k|}_{\mathbb{1}} \rho_A\, |i\rangle = \mathrm{tr}[O_A \rho_A]$$

$$|\psi\rangle = \sum_k s_k |\alpha_k\rangle |\beta_k\rangle$$

$\Rightarrow \rho_A = \sum_k s_k^2\, |\alpha_k\rangle\langle\alpha_k|$ Eigenvalues of ρ_A.

More entanglement ⇒ more uncertainty in ρ_A!
(although <u>global</u> state pure!)

How can we <u>measure</u> entanglement <u>beyond</u> Schm.-coeff.?

E.g.: $|\psi_1\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ vs.

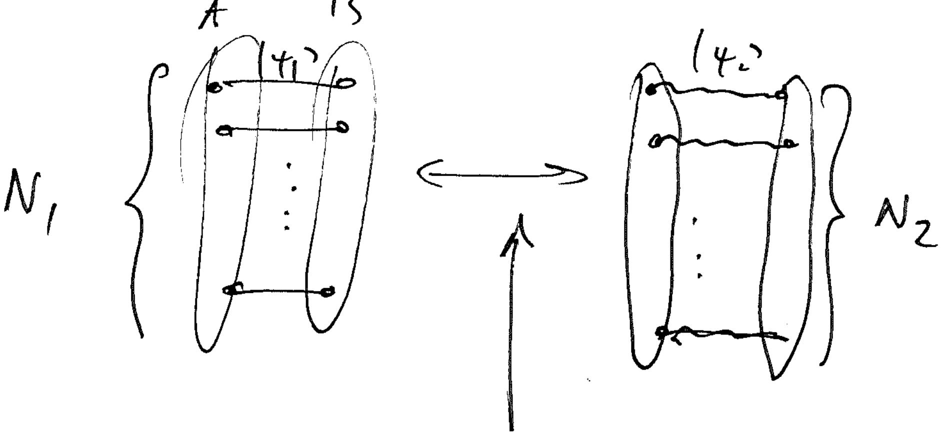
$|\psi_2\rangle = \epsilon\,|01\rangle + \sqrt{1-\epsilon^2}\,|10\rangle$ which is more entangled?

<u>Interconversion:</u> Can A&B convert $|\psi_1\rangle \Leftrightarrow |\psi_2\rangle$ w/ LOCC,

and ~~at~~ with which probability?

a) for single copies: $|\psi_1\rangle \overset{\text{different rates}}{\rightleftarrows} |\psi_2\rangle$ "Majorization"

b) Asymptotic scenario:



<u>reversible conversion</u> possible!

<u>Rate</u>: $S(\rho_A^1)\cdot N_1 = S(\rho_A^2)\cdot N_2$

Here: $\boxed{S(\rho) = -\rho\log\rho}$, ← von Neumann entropy

Shannon entropy

$= -\sum \lambda_i \log \lambda_i$, w/ λ_i eigenvals of ρ!

In particular: $|\psi_1\rangle = (|01\rangle + |10\rangle)/\sqrt{2} \Rightarrow \rho_A = \begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix}$

$\Longrightarrow S(\rho_A) = 1$ "1 ebit"

$$\boxed{\text{Entropy} \equiv \text{Entanglement}}$$

uncertainty about subsystem when total system has ~~(for certain global state)~~ no uncertainty!

Note: there exist other measures of uncertainty (entropies), e.g. the Rényi entropies

$$S_\alpha(\rho) = \frac{\log \operatorname{tr} \rho^\alpha}{1-\alpha}.$$

Note: $S_0(\rho) = \log \operatorname{rank} \rho$

$$\lim_{\alpha \to 1} S_\alpha(\rho) = S(\rho)$$

Ent. of multipartite states:

$$|\psi\rangle \in \underbrace{\mathbb{C}^d \otimes \mathbb{C}^d \otimes}_{\substack{A \\ L \text{ sites}}} \underbrace{\dots \otimes \mathbb{C}^d}_{\substack{B \\ N-L \text{ sites.}}}$$

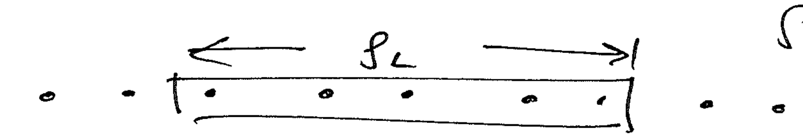
$\rho_A = \operatorname{tr}_B |\psi\rangle\langle\psi| \Rightarrow$ $\underline{S(\rho_A) : \text{ent. betw. } A \,\&\, B}$

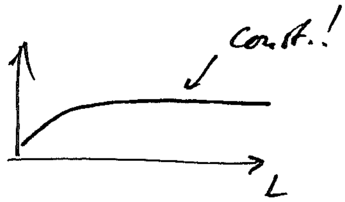
For a <u>random state</u>, and L suff. smaller than $N/2$:

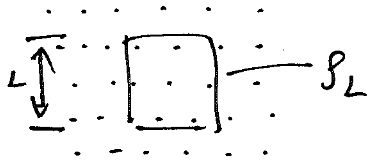
$S(\rho_A)$ is essentially maximal, $S(\rho_A) \sim L \log d$

$$\rho_A \approx \frac{\mathbb{1}}{d^L} \; !$$

For <u>ground states</u> of local Hamiltonians:

<u>1D</u>: 

$S(\rho_L)$: 

<u>2D</u>: 

: $S(\rho_L) \sim L$!

<u>Surface</u> scaling (vs. Volume)

$\Rightarrow$ area law