Lecture "Analytical and Numerical Methods for Quantum Many-Body Systems from a Quantum Information Perspective" — Exercise Sheet #1

— The problem sheets will not be graded. The idea is that they help to improve the understanding of the subject and to find out where there are difficulties, which can then be discussed in the exercises. To nevertheless give some "priority" to the problems on the sheet so you know on which to focus, I would think that 6 and 7 are particularly important (but also potentially more difficult), and 1 and 3 are very optional. —

1. Generalize the proof of the Singular Value Decomposition given in the lecture to the case of non-square matrices, and matrices M where both $MM^\dagger$ and $M^\dagger M$ do not have full rank.

2. Find the Schmidt decompositions and the reduced density matrices for the following states on two qubits. You can try to find the Schmidt decomposition either using the singular value decomposition or the reduced density matrices.

$$\frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}} \tag{1}$$
$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \tag{2}$$
$$\frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2} \tag{3}$$
$$\frac{|00\rangle + 2|01\rangle + 2|10\rangle - \alpha|11\rangle}{\sqrt{9+\alpha^2}} \tag{4}$$

   It is completely fine to use a computer algebra system, in particular for the last one.

3. Check that the Rényi entropy
$$S_\alpha(\rho) = \frac{\log(\operatorname{tr}\rho^\alpha)}{1-\alpha}$$
   converges to the von Neumann entropy
$$S(\rho) = -\operatorname{tr}(\rho\log\rho)$$
   for $\alpha \to 1$.

4. Consider a bipartite state $|\psi\rangle_{AB}$ shared by Alice and Bob, and let Alice and Bob apply a linear map X_A and Y_B to their part of the system, respectively. (It might not be possible to implement these maps deterministically, but they can always be implemented probabilistically, and if Alice and Bob can communicate, they can always determine if both maps have been applied successfully.) The joint map applied to $|\Psi\rangle_{AB}$ is then $X_A \otimes Y_B$, resulting in the state $|\Phi\rangle_{AB} = (X_A \otimes Y_B)|\Psi\rangle_{AB}$. Show that the Schmidt rank (i.e., the number of non-zero Schmidt coefficients) of $|\Phi\rangle_{AB}$ cannot be larger than the Schmidt rank of $|\Psi\rangle_{AB}$. (I.e.: Show that the Schmidt rank cannot be increased by local operations.)

5. Show that the four Bell states

$$|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{5}$$
$$|\Phi^-\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{6}$$
$$|\Psi^+\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{7}$$
$$|\Psi^-\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{8}$$

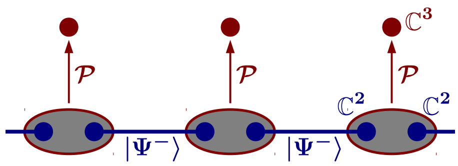
   can be converted into each other by a unitary U acting on *one* side only (i.e., as $U \otimes \mathbb{I}$).

6. The AKLT state is constructed by placing singlet states $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ on a chain with periodic boundary conditions (PBC), and applying a linear map $\mathcal{P} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^3$ which projects onto the symmetric (spin-1) subspace, i.e.,

$$\mathcal{P} = |1\rangle\langle 11| + |0\rangle\frac{\langle 01| + \langle 10|}{\sqrt{2}} + |-1\rangle\langle 00|\ ,$$

see the figure below. Find the (translational invariant) MPS representation of the AKLT state, i.e., the matrices A^{-1}, A^0, and A^1. For this, it is essential to note that the singlet bond can be replaced by a normal $|00\rangle + |11\rangle$ bond, by appropriately modifying $\mathcal{P}$ (see the problem 5).

Try also to rewrite the MPS representation using the local basis states $|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |-1\rangle)$ and $|0\rangle$.



7. Find MPS representation on a chain of qubits of length N (either with periodic boundary conditions (PBC) or open boundary conditions (OBC)) for the following states
   - The GHZ state $|0\ldots0\rangle + |1\ldots1\rangle$
   - The W state $|100\ldots0\rangle + |010\ldots0\rangle + |001\ldots0\rangle + \cdots + |000\ldots1\rangle$
   - **(hard)** The *cluster state*, which is created by initializing all qubits in the $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ state and then applying a controlled phase gate

$$\mathrm{CPHASE} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

     to all nearest neighbors. (The CPHASE gates commute, so the order in which they are applied is not important.)

     To find the MPS representation, first consider only the CPHASE gate and try to express it using local maps on the two qubits together with shared entanglement – for instance the state $|00\rangle + |01\rangle + |10\rangle - |11\rangle$, which already resembles the structure of the phase gate. Alternatively, you can try to express the CPHASE gate in the tensor network picture as a tensor with two inputs and two outputs, which is then decomposed into two tensors acting on the two qubits, with some shared index which implements the control condition for the phase.