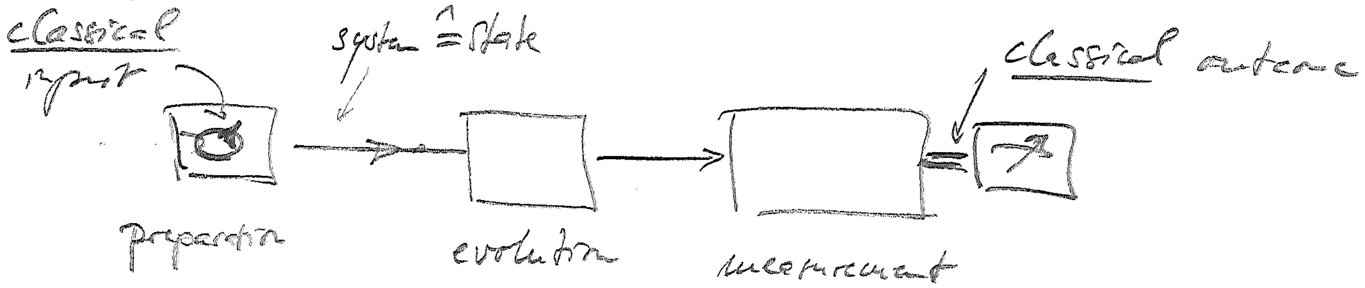
II. The formalism: States, measurements, and evolution

1. Axioms ("pure" version):

- Setup in Q.M.:



- State of the system = description of our knowledge of system (= prediction of outcomes ...)

- States are vectors in Hilbert spaces!

  (More precisely: rays.)

Hilbert space: $\mathcal{H}$ is a Hilbert space iff

i) $\mathcal{H}$ is a vector space over $\mathbb{C}$. We denote vectors by $|v\rangle \in \mathcal{H}$.

ii) $\mathcal{H}$ has a scalar product $\langle v|w\rangle$ s.t.

- $\langle v|v\rangle > 0$ if $|v\rangle \neq 0$
- $\langle v|a_1 w_1 + a_2 w_2\rangle = a_1\langle v|w_1\rangle + a_2\langle v|w_2\rangle$ for $a_1, a_2 \in \mathbb{C}$.

- $\langle v|w\rangle = \langle w|v\rangle^*$

iii) $\mathcal{H}$ is <u>complete</u> in the norm

$$\|v\| = \sqrt{\langle v|v\rangle}$$

(<u>Note</u>: The property iii) – completeness – is always satisfied in finite dimensions – which we focus on – so we can safely ignore it most of the time)

<u>Ket/bra notation</u>:

$|v\rangle$ denotes vectors ("column vectors"),

$\langle v|$ denotes their duals ("row vectors")

$$\langle v| \cdot |w\rangle \equiv \langle v|w\rangle$$

<u>Basis</u>:

We can expand vectors in an orthonormal basis

$$|e_i\rangle \equiv |i\rangle, \qquad \langle i|j\rangle = \delta_{ij}$$

$$\Rightarrow |v\rangle = \sum_{i=1}^{d} v_i |i\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}$$

$$\langle v| = \sum_i v_i^* \langle i| = (v_1^*, \dots, v_d^*)$$

Quantum systems $\longleftrightarrow$ Hilbert space $\mathcal{H}$

State of system $\longrightarrow$ vector $|\psi\rangle \in \mathcal{H}$ with $\| |\psi\rangle \| = 1$.

<u>Note</u>: States $|\psi\rangle$ and $c|\psi\rangle$ describe the same state.

Linear operators:

$M : \mathcal{H} \longrightarrow \mathcal{H}$ is linear:

$$M(|v\rangle + \alpha|w\rangle) = M(|v\rangle) + \alpha M(|w\rangle) \quad \text{for } \alpha \in \mathbb{C}.$$

Write $M(|v\rangle) = M|v\rangle$.

Expansion in ONB:

$$M = \left(\sum_i^d |i\rangle\langle i|\right) M \left(\sum_j^d |j\rangle\langle j|\right)$$

$$= \sum_{ij}^d M_{ij}\, |i\rangle\langle j| = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1d} \\ \vdots & & & \\ M_{d1} & \cdots & \cdots & M_{dd} \end{pmatrix}$$

with $M_{ij} = \langle i|M|j\rangle$.

∘ Evolution: unitary transformation $U : \mathcal{H} \to \mathcal{H}$.

$$|\psi\rangle \longmapsto U|\psi\rangle.$$

Unitary: $\underbrace{(U|\psi\rangle)^\dagger}_{= \langle\psi|U^\dagger} \cdot (U|\phi\rangle) = \langle\psi|\phi\rangle$, i.e., angle-preserving.

In basis: $(U^\dagger)_{ij} = U^{*}_{ji}$;

Unitarity $\Longleftrightarrow U^\dagger U = \mathbb{1}$.

- Hamiltonians:

Unitary evolution is generated by a Hamiltonian H, $H = H^\dagger$:

Infinitesimal $\mathcal{U} \approx \mathbb{1} + \delta t \cdot K$:

$$\mathbb{1} = \mathcal{U}^\dagger \mathcal{U} = (\mathbb{1} + \delta t K^\dagger)(\mathbb{1} + \delta t K)$$

$$= \mathbb{1} + \delta t \underbrace{(K + K^\dagger)}_{\stackrel{!}{=} 0} + O(\delta t^2)$$

$\Rightarrow K = -K^\dagger$. With $H = iK$: $H = H^\dagger$.

Unitaries $\leftrightarrow$ evolve with Hamiltonian $H(t)$,

$$\mathcal{U}(t) = e^{iHt} \quad \text{or} \quad \mathcal{U}(t) = \mathcal{T} e^{\int iH(t) dt}.$$

- Measurement:

Observable quantities = $\underbrace{\text{hermitian}}_{\text{i.e. } A = A^\dagger}$ operators A

Eigenvalue decomposition: eigenbasis of A!

$$A = \sum_{n=1}^{d} a_n |n\rangle\langle n| = \sum a_n E_n$$

spectral projector: $|n\rangle\langle n|$, or $\sum_{n \text{ dege.}} |n\rangle\langle n|$ for degen. eigenvalues)

Measurement of A on state $|\psi\rangle$:

Outcome a_n with probability $|\langle n|\psi\rangle|^2$

or $\langle\psi|E_n|\psi\rangle = \| E_n|\psi\rangle\|^2$.

State after measurement:

$$|\psi_n\rangle = \frac{E_n|\psi\rangle}{\| E_n|\psi\rangle\|}$$

Expectation value (= average meas. outcome):

$$\langle\psi|A|\psi\rangle = \sum a_n \langle\psi|E_n|\psi\rangle.$$

Axioms: • States are vectors $|\psi\rangle \in \mathcal{H}$, $\||\psi\rangle\| = 1$,
with $|\psi\rangle \sim c|\psi\rangle$ the same state (rays).

• Evolution $\mathcal{U}: |\psi\rangle \mapsto U|\psi\rangle$ is unitary.

• Measurements $A = \sum a_n E_n$ ← proj. act as

$$|\psi\rangle \mapsto |\psi_n\rangle = \frac{E_n|\psi\rangle}{\|E_n|\psi\rangle\|} \text{ w/ prob. } \|E_n|\psi\rangle\|^2.$$

- Composite systems:
  - We consider a system with two separate parts ("subsystems") A (= Alice) and B (= Bob).

A — underlying Hilbert space $\mathcal{H}_A$

B — underlying Hilbert space $\mathcal{H}_B$

$\Rightarrow$ Joint system: Underlying Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

$\rightarrow$ State described by $|\psi\rangle_{AB} \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

General form of vector in $\mathcal{H}_A \otimes \mathcal{H}_B$?

$|i\rangle_A$ ONB of $\mathcal{H}_A$, $i = 1, \dots, d_A$

$|j\rangle_B$ ONB of $\mathcal{H}_B$, $j = 1, \dots, d_B$.

$\Rightarrow$ $|i\rangle_A \otimes |j\rangle_B = |i\rangle_A |j\rangle_B = |i,j\rangle_{AB} = |ij\rangle_{AB}$

basis of $\mathcal{H}_A \otimes \mathcal{H}_B$; $i = 1, \dots, d_A$; $j = 1, \dots, d_B$

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$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$|\psi\rangle = \sum c_{ij} |i\rangle_A |j\rangle_B$: $d_A \times d_B$ - dimensional.

$= (c_{11}, c_{12}, c_{13} \ldots c_{1d_B}, c_{21}, c_{22}, \ldots, c_{d_A d_B})^T$.

What happens if Alice acts w/ M_A on her system, and Bob with N_B on his?

(<u>Note:</u> M_A, N_B could be unitaries, measurements, meas. projectors E_u, ... — or even the <u>trivial</u> action $N_B = \mathbb{1}_B$ if only Alice acts on her system.)

Total action given by $M_A \otimes N_B$:

$$(M_A \otimes N_B)(|i\rangle_A \otimes |j\rangle_B) = (M_A |i\rangle_A) \otimes (N_B |j\rangle_B)$$

Matrix elements:

$$\langle i_A i_B | M_A \otimes N_B | j_A j_B \rangle = \langle i_A | M_A | j_A \rangle \langle i_B | N_B | j_B \rangle$$

$$\|$$

$$(M_A \otimes N_B)_{(i_A i_B)(j_A j_B)} = (M_A)_{i_A j_A} (N_B)_{i_B j_B}$$

Matrix notation:

$$(\Pi_A \otimes N_B) = \left(\begin{array}{c|c} (\Pi_A)_{11} N_B & (\Pi_A)_{12} N_B \quad \cdots \\ \hline (\Pi_A)_{12} \cdot N_B & \\ \vdots & \ddots \end{array}\right)$$

Examples:

Qubits: $\mathcal{H} = \mathbb{C}^2$; "computational basis" $|0\rangle, |1\rangle$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad , \; |\alpha|^2 + |\beta|^2 = 1$$

Observable $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underbrace{\overbrace{|0\rangle\langle 0|}^{E_0} - \overbrace{|1\rangle\langle 1|}^{E_1}}_{\text{eigenbasis!}}$

$\rightarrow$ eigenvalues $+1$ w/ eigenvector $|0\rangle$

-1 w/ eigenvector $|1\rangle$

Measurement: $\dfrac{E_0|\psi\rangle}{\|E_0|\psi\rangle\|} = |0\rangle$ w. prob. $\|E_0|\psi\rangle\|^2 = |\alpha|^2$

$\dfrac{E_1|\psi\rangle}{\|E_1|\psi\rangle\|} = |1\rangle$ w. prob. $\|E_1|\psi\rangle\|^2 = |\beta|^2$

Observable $X = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underbrace{\overbrace{|+\rangle\langle +|}^{E_+} - \overbrace{|-\rangle\langle -|}^{E_-}}$

with $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ eigensystem

Measurement:

$$\frac{E_\pm|\psi\rangle}{\|E_\pm|\psi\rangle\|} = |\pm\rangle \text{ w. prob. } |\langle\pm|(\alpha|0\rangle+\beta|1\rangle)|^2 = \frac{|\alpha\pm\beta|^2}{2}$$

Evolution:

Ham. $Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Evolve for time $\pi/4$ ($\hbar = 1$.)

$$U = e^{-i\pi/4 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}. \quad \text{Use } Y^2 = \mathbb{1}$$

$$= \cos(\pi/4)\,\mathbb{1} - i\sin(\pi/4)\,Y$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} +1 & +1 \\ -1 & +1 \end{pmatrix}.$$

$$U|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\cdot(\alpha|0\rangle+\beta|1\rangle)$$

$$= \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|0\rangle - \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|1\rangle$$

Meas. in $|0\rangle, |1\rangle$ (Z) basis:

0 w/ prob. $\frac{|\alpha+\beta|^2}{2}$

1 w/ prob. $\frac{|\alpha-\beta|^2}{2}$

$\Rightarrow$ Y transforms (almost...) between X and Z basis!
(i.e.: X meas. can be realized as evolution + Z meas.)

<u>Note:</u> Seq. meas. of $Z \to X \to Z$ will give a <u>random outcome</u> for 2nd Z meas.

$\Rightarrow$ X and Z cannot be measured simultaneously (⇒ uncertainty!)

<u>Measurement on a bipartite state:</u>

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Alice measures Z, Bob measures Z:

Projectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

$\Rightarrow$ 01 and 10 w. prob $1/2$

Alice measures X, Bob meas. X:

Projectors $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$:

(use $\langle +|0\rangle = \langle +|1\rangle = \langle -|0\rangle = 1/\sqrt{2}$; $\langle -|1\rangle = -1/\sqrt{2}$)

$$|\langle ++|\psi\rangle|^2 = 0$$

$$|\langle +-|\psi\rangle|^2 = \left|-\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\langle -+|\psi\rangle|^2 = \dots = \frac{1}{2}$$

$$|\langle --|\psi\rangle|^2 = \dots = 0$$

$\Rightarrow$ anticorrelation

In fact, anti-corr in <u>all</u> basis (HW)