# Exercise Sheet 5

*Quantum Information*

**To be returned no later than May 21, 2015**

(20 points) **Problem 1: Entropy.**

1) Show that entropies are additive, $S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$.

2) The conditional quantum entropy $S(A|B)$ of a bipartite quantum state $\rho_{AB}$ is defined as

$$S(A|B) := S(\rho) - S(\mathrm{Tr}_A(\rho)) \ .$$

The corresponding classical quantity $H(X|Y) := -\sum_y p(y) \sum_x p(x|y) \log p(x|y)$ measures how much we learn on average about $X$ by learning $Y$.

Find a state $\rho_{AB}$ for which $S(A|B)$ is negative, something which which cannot happen for the classical quantity. (This is, in quantum mechanics, there can be states for which we can be more certain about the joint state than about its individual parts. If we measure B, we thereby acquire "negative information", since we destroy entanglement which could otherwise have been used as a resource for e.g. teleportation. See also http://goo.gl/9vsWGt).

3) Calculate $S(A|B)$ for the maximally entangled state $|\Phi\rangle = \frac{1}{\sqrt{D}} \sum_{j=1}^{D} |j\rangle_A |j\rangle_B$, the maximally correlated state $\bar{\Phi}_{AB} = \frac{1}{D} \sum_{j=1}^{D} |j\rangle \langle j|_A \otimes |j\rangle \langle j|_B$, and a product state $\rho = \rho^A \otimes \rho^B$.

(15 points) **Problem 2: Decay of entanglement.**

Consider a Bell state $\rho_2 = |\Phi^+\rangle \langle \Phi^+|$, where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Superposition states like $\rho_2$ generally are not stable, but decay over time. A typical evolution is that the populations (i.e., the diagonal elements) becom equal, while the off-diagonal elements decay to zero. Suppose that the state evolves as

$$\rho_2(t) = p_+ |00\rangle \langle 00| + p_- |01\rangle \langle 01| + p_- |10\rangle \langle 10| + p_+ |11\rangle \langle 11| + e^{-t/T_2}/2 \, |00\rangle \langle 11| + e^{-t/T_2}/2 \, |11\rangle \langle 00| \,,$$

with $p_\pm = \frac{1}{4}(1 \pm e^{-t/T_1})$.

For sufficiently long times, this state tends to $\lim_{t\to\infty} \rho_2(t) = \frac{1}{4}\mathbb{I}$, the maximally mixed state.

1) Write a matrix form of state $\rho_2(t)$.

2) Take its partial transpose $\rho_2(t)^{T_B}$ and write its matrix form.

3) Calculate the eigenvalues of $\rho_2(t)^{T_B}$.

(You may use a computer algebra system, though it should not be necessary.)

4) Sketch how the eigenvalues change over time for $T_1 = T_2 = 1$ (clearly mark the asymptotic limit).

5) Is state $\rho_2(t = 0)$ is entangled or separable? Find time after which state $\rho_2(t)$ becomes separable.

(15 points) **Problem 3: Bell inequalities and witnesses.**

The CHSH operator

$$C = \vec{n}_1\vec{\sigma} \otimes \vec{n}_0\vec{\sigma} + \vec{n}_1\vec{\sigma} \otimes \vec{n}_2\vec{\sigma} + \vec{n}_3\vec{\sigma} \otimes \vec{n}_2\vec{\sigma} - \vec{n}_3\vec{\sigma} \otimes \vec{n}_0\vec{\sigma}$$

with $\vec{n}_k = (\cos(k\pi/4), 0, \sin(k\pi/4))$ has the property that $|\mathrm{tr}[C\rho]| \leq 2$ for all $\rho$ which describe a local hidden variable (LHV) model. Note that any separable state $\rho = \sum p_i \rho_i^A \otimes \rho_i^B$ describes a LHV model.

1) Use $C$ to construct an entanglement witness $W$. Provide an explicit form of the witness. (You may use that all separable states are LHV models to prove that $\mathrm{tr}[W\rho] \geq 0$.)

2) In which range of $\lambda$ does this witness detect Werner states $\rho(\lambda) = \lambda \left|\Psi^-\right\rangle \left\langle\Psi^-\right| + \frac{1-\lambda}{4}\mathbb{I}$, with $\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$? How does it compare to the entanglement witness $W = \mathbb{F}$ discussed in the lecture?

(25 points) **Problem 4: Witnesses and reduction criterion.**

Consider $W = \mathbb{I} - d\,|\Omega\rangle\langle\Omega|$, with $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i,i\rangle$.

1) Show that $\mathrm{tr}[W\rho] \geq 0$ for separable states $\rho$, i.e., $W$ is an entanglement witness. (Some results of Sheet 4, Problem 5 might be helpful!)

2) Consider the family

$$\rho_{\mathrm{iso}}(\lambda) = \lambda \frac{\mathbb{I}}{d^2} + (1-\lambda)\,|\Omega\rangle\langle\Omega|$$

of *isotropic states*. In which range of $\lambda$ is $\rho_{\mathrm{iso}}(\lambda) \geq 0$? In which range of $\lambda$ does $W$ detect that $\rho_{\mathrm{iso}}(\lambda)$ is entangled?

3) Consider the case $d = 2$. What does $W$ do on the antisymmetric state $\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$?

4) Derive the positive map $\Lambda$ corresponding to the witness $W$. Prove directly that it is indeed a positive map.

5) In which range of $\lambda$ does $\Lambda$ detect that $\rho_{\mathrm{iso}}(\lambda)$ is entangled? What does $\Lambda$ do on the antisymmetric state?

(25 points) **Problem 5: Partial transposition and decomposible maps**

1) Show that $\rho^{T_A} \geq 0$ if and only if $\rho^{T_B} \geq 0$. Does it also hold that $\rho^{T_A} = \rho^{T_B}$?

2) Show that $(\Lambda \otimes \mathbb{I})(\rho^{T_B}) = \left[(\Lambda \otimes \mathbb{I})(\rho)\right]^{T_B}$.

2) A positive map $\Lambda$ is called decomposable if it can be written as $\Lambda(\rho) = \mathcal{D}(\rho) + \mathcal{E}(\rho^T)$, with $\mathcal{D}$ and $\mathcal{E}$ completely positive maps. Prove that whenever a decomposible map $\Lambda$ detects an entangled state $\rho$, that state is also detected by the PPT criterion.

3) Show that if $\Lambda(\rho)$ is decomposable, the corresponding witness $W = ((\Lambda \otimes I)(|\Omega\rangle\langle\Omega|))^T$ is of the form $W = P + Q^{T_B}$, with $P, Q \geq 0$.

4) Show that the witness from problem 4 is of this form, i.e., the corresponding map $\Lambda$ is decomposable. (*Hint* (to be proven): $d\,|\Omega\rangle\langle\Omega| = \mathbb{F}^{T_B}$).

5) Combine this to make a hierarchy of entanglement criteria (i.e., to sort them by their strength), including the witness $W$ and map $\Lambda$ from problem 4 the witness $\mathbb{F}$, and the PPT criterion.